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Rotating Black Holes in Gauged Supergravities; Thermodynamics, Supersymmetric Limits, Topological Solitons and Time Machines

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ABSTRACT

We study the thermodynamics of the recently-discovered non-extremal charged rotating black holes of gauged supergravities in five, seven and four dimensions, obtaining energies, angular momenta and charges that are consistent with the first law of thermodynamics. We obtain their supersymmetric limits by using these expressions together with an analysis of the AdS superalgebras including R-charges. We give a general discussion of the global structure of such solutions, and apply it in the various cases. We obtain new regular supersymmetric black holes in seven and four dimensions, as well as reproducing known examples in five and four dimensions. We also obtain new supersymmetric non-singular topological solitons in five and seven dimensions. The rest of the supersymmetric solutions either have naked singularities or naked time machines. The latter can be rendered non-singular if the asymptotic time is periodic. This leads to a new type of quantum consistency condition, which we call a *Josephson quantisation condition*. Finally, we discuss some aspects of rotating black holes in Gödel universe backgrounds.

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Contents

1 Introduction

With the development of the AdS/CFT correspondence [1, 2], it has become important to investigate backgrounds in gauged supergravities describing charged black holes. Of particular interest are five-dimensional gauged supergravity, in the context of the type IIB string, and the gauged supergravities in four dimensions and seven dimensions, in the context of M-theory. Supersymmetric black holes in the AdS background play a particularly important role. Typically, it turns out that these are singular unless they are in addition rotating, and so it becomes essential to study rotating charged black holes in gauged supergravity.

A point of special interest is the occurrence of closed timelike curves in supersymmetric solutions, including solutions with a large amount of supersymmetry. This raises the issue of whether Chronology Protection may be achieved by some stringy or quantum mechanical consistency condition. For example is it associated with a breakdown of unitarity of the boundary field theory? It is well known that the Kerr solution has closed timelike curves and it remains an open question whether they play a role in realistic models of gravitational collapse. Supergravity solutions with closed timelike curves offer the opportunity of studying these questions within a well defined and controlled theoretical context.

The general solutions for non-extremal charged rotating black holes in ungauged supergravities have been obtained, in five dimensions [3], four dimensions [4], and then in other dimensions [5, 6]. The thermodynamics and grey body factors of general five and four dimensional solutions we studied in [7] and [8], respectively. The global properties of a special case in five dimensions [9], where the three electric charges of the general solution are set equal, have been studied in considerable detail [10]. They can in fact be constructed by a rather mechanical solution-generating procedure, involving the use of global symmetries of the ungauged supergravity theories. In this construction, one begins with the standard uncharged rotating black hole solutions, and introduces the charges by making global symmetry transformations. The extremal limits of these charged rotating black holes provide supersymmetric BPS configurations.

Constructing the analogous charged rotating solutions in gauged supergravity is a much more challenging problem. The general uncharged AdS black hole solutions were found in four dimensions in [11], in five dimensions in [12], and in all dimensions $D \geq 6$ in [13, 14]. However, in the gauged supergravities one no longer has global symmetries that can be used in order to introduce charges. Instead, there is little option but to resort to more "brute force" methods for solving the supergravity equations. By such methods, various non-extremal charged rotating black hole solutions have been constructed in gauged

supergravities in four, five and seven dimensions. The simplest case, obtained long ago in [15], is the Kerr-Newman-AdS black hole in four dimensions, which as a solution of the Einstein-Maxwell system with a cosmological constant, can be viewed also as a solution in four-dimensional gauged $\mathcal{N} = 2$ supergravity.

Recently, non-extremal charged rotating black hole solutions in five-dimensional gauged supergravity have been constructed. In all of these, the problem was simplified by taking the two rotation parameters of a general five-dimensional rotating black hole to be equal. First, the solution in five-dimensional gauged $\mathcal{N} = 2$ (minimal) supergravity was found [16]. This was generalised to the case of maximal gauged supergravity, with the black holes carrying three independent electric charges, in [17]. Setting the three charges equal reduces to the case studied previously in [16].

Generalisations of the Kerr-Newman-AdS black holes in four dimensions were then found [18], which can be viewed as solutions in gauged $\mathcal{N} = 4$ supergravity, with two independent electric (or magnetic) charges. If these are set equal, the solutions reduce to the Kerr-Newman-AdS black holes. In terms of gauged $\mathcal{N} = 8$ supergravity, the solutions in [18] correspond to 4-charge black holes, in which the charges are set pairwise equal.

In seven dimensions, non-extremal charged rotating black hole solutions in the maximal gauged $\mathcal{N} = 4$ supergravity were obtained in [19], with two independent electric charge parameters. Again, the problem was simplified by taking the three rotation parameters of a general seven-dimensional black hole to be equal.

Certain supersymmetric rotating black holes in gauged supergravities have also been previously obtained. Specifically, in four dimensions it was shown in [20] that one can take a BPS limit of the Kerr-Newman-AdS black hole. In fact in general this BPS limit is not a black hole at all, since it lacks an event horizon and thus it has a naked singularity. However, as shown in [20], if a certain relation between the angular momentum and the charge holds, one obtains a genuine supersymmetric rotating black hole.

In five dimensions, supersymmetric solutions including black holes were obtained in gauged supergravity by a direct approach, in which the supersymmetry was imposed from the outset. Examples included solutions found by Klemm and Sabra [21, 22], which contain naked singularities or closed timelike curves, and solutions found by Gutowski and Reall [23, 24], which have regular horizons.

The plan of the paper is as follows. We begin in section 2 by recalling some basic definitions associated with horizons and closed timelike curves (CTC's), using the BMPV black hole [9] as an illustrative example. We pay special attention to the asymptotically

AdS case and the properties of Killing vectors arising from Killing spinors. We show how the metrics we are considering allow a complete integration of the geodesics using Hamilton-Jacobi theory, and this allows us to give a simple physical derivation of the second law of thermodynamics for black holes. At the end of section 2, we give a derivation of a new type of quantisation condition that arises when ensuring that solutions are globally well-defined. This new condition arises only if the time coordinate is periodic. Because an analogous condition arises when considering the time-dependent Josephson effect, we propose calling this new quantisation condition the Josephson quantisation condition. (See [25] for a related discussion of the standard time-dependent Josephson effect using just the general ideas of gauge-invariance rather than a particular microscopic model as in Josephson's original derivation.)

In the following three sections, we study rotating black holes in $D = 5$, $D = 7$ and $D = 4$ gauged supergravities, and their supersymmetric limits. We find many interesting such limits, but the most important outcomes are new regular supersymmetric rotating black holes in seven dimensions and in four dimensions, and regular topological solitons in $D = 5$ and $D = 7$. In detail, these sections are as follows.

Section 3 is devoted to a study of rotating black holes in five-dimensional gauged supergravity. Starting from a general class of charged rotating black holes found in [16, 17], we obtain expressions for their mass, angular momentum, charges, temperature, angular velocity and electrostatic potentials, and we confirm that the first law of thermodynamics holds. We then turn to a consideration of the constraints placed by supersymmetry. We outline the derivation of a supersymmetric bound, which allows us to deduce how the parameters of the solutions must be chosen in order to obtain supersymmetric configurations. We then investigate the global structure and regularity of these supersymmetric configurations. The classification is quite rich. Roughly speaking, the supersymmetric solutions fall into two classes, which we shall call A and B.

The solutions in class A generically preserve $\frac{1}{2}$ of the $\mathcal{N} = 2$ supersymmetry. Their classification resembles in some respects that of the BMPV black holes. For certain ranges of the parameters, the solutions are free of CTC's, but there is a naked singularity. For other ranges of the parameters, CTC's are present. Inside the region of CTC's (*i.e.* inside the time machine), there is what formally looks like an event horizon, with imaginary temperature and area. However, this is just a coordinate singularity; the spacetime can be closed off to give a non-singular metric at this point, at the expense of making a *real* periodic identification of the time coordinate. We refer to the "horizon" in this case as a pseudo-

horizon. The identification of the time coordinate is the explanation of the previously puzzling observation that the temperature becomes imaginary in this case. The resulting spacetime resembles the repulson case of the BMPV solution. However, in the BMPV case, there was no need to identify the time periodically. In the present case, the identification of the time coordinate requires that we impose a Josephson-type condition, in order that the fermions in the gauged supergravity be well defined.

There is an intermediate case of critical rotation, which is perhaps the most interesting, and appears to be an entirely new type of *topological soliton* solution. It is a completely non-singular globally stationary spacetime, with no horizon, defined on the product of time with a spatial manifold having the topology of an $\mathbb{R}^2$ bundle over S^2. The metric is globally hyperbolic, and completely free of CTC's. It differs from the recently-discovered bubble or droplet solutions, which are globally static, charged but non-rotating, and topologically trivial [26, 27]. By contrast, this new solution is topologically non-trivial, and carries angular momentum. We have checked that the new solution has zero entropy, but satisfies the first law of thermodynamics with the TdS term making no contribution. In some cases, the soliton solutions are quantum-mechanically consistent.

The five-dimensional supersymmetric solutions of type B generically preserve $\frac{1}{4}$ of the $\mathcal{N}=2$ supersymmetry, and generically have CTC's. There is either a naked singularity inside the region of CTC's (*i.e.* inside the time machine), or there is a formal horizon inside the time machine, at which the the spacetime closes off to give a geodesically-complete repulson type solution with periodic time. In addition there are two further special cases. In one, the CTC's are confined inside a Killing horizon to give a supersymmetric black hole that is regular outside and on the horizon. This is the solution obtained in [23, 24]. The other special case arises when the horizon and the boundary of the time machine coincide. This is a regular soliton solution, with no horizons and no CTC's, analogous to the soliton solution we have found amongst the type A solutions.

Section 4 is devoted to a study of rotating black holes in seven-dimensional gauged supergravity. We obtain the mass, angular momentum, charges, temperature, angular velocity and electrostatic potentials of a general class of 2-charge black holes constructed in [19], and confirm that the first law of thermodynamics is satisfied. The supersymmetric cases are again investigated using the supersymmetry bound, which allows us to determine the parameters for which supersymmetry is possible. Again, there are two classes, which we shall refer to as class A and class B. Solutions in class A generically preserve $\frac{3}{8}$ of the $\mathcal{N}=2$ supersymmetry. If the angular momentum is positive, all such solutions have naked

singularities inside a time machine. This continues to be true if the angular momentum is negative, up to a critical value below which we have a repulson-like behaviour inside a time machine. The critical case arises when the boundary of the repulson coincides with the boundary of the time machine, and again we obtain a completely non-singular topological soliton type solution.

The seven-dimensional supersymmetric solutions of type B generically preserve $\frac{1}{8}$ of the $\mathcal{N} = 2$ supersymmetry. Generically, these solutions have naked singularities or naked CTC's. However, we find for special choices of the parameters that we can obtain completely regular black holes, with CTC's occurring only inside the horizon. These new solutions are seven-dimensional analogues of the five-dimensional rotating black holes found in [23, 24]. We also find a non-singular topological soliton for another special choice of the parameters.

Section 5 is devoted to rotating black holes in four-dimensional gauged supergravity. We obtain the thermodynamic quantities for the general class of 4-charge black holes (with charges set pairwise equal) that were obtained in [18], and we confirm that the first law is satisfied. We obtain the conditions under which the solutions are supersymmetric, and find new regular supersymmetric rotating black holes.

In section 6, we apply some of the ideas and concepts developed earlier to some related solutions of current interest in five-dimensional ungauged supergravity. These solutions describe black holes in a Gödel universe. The background is supersymmetric, but contains CTC's. We show that if the black hole is sufficiently large that the horizon formally enters the region of CTC's, then a repulson phenomenon takes place in which the space compactifies, and time must be identified periodically. We conclude section 6 with a further discussion of the homogeneous Gödel background spacetime itself. We show that if one wishes to make identifications to compactify the spatial sections, it is necessary to make the time periodic also, thereby obtaining a complete non-singular compact spacetime.

2 Global Considerations

2.1 Causality

Consider a Lorentzian metric on some spacetime M with two commuting Killing vector fields ∂_t and ∂_ψ Assume also that $\psi \in (0, \frac{4\pi}{k}]$, $k \in \mathbb{N}$. The metric may be given the form

$$ds^2 = g_{tt} dt^2 + 2 g_{t\psi} (d\psi + \omega) dt + g_{\psi\psi} (d\psi + \omega)^2 + g_{ab} dy^a dy^b \,. \tag{2.1}$$

We have assumed that all $dtdy^a$ terms vanish and that all $d\psi dy^a$ terms may be absorbed in ω which is a one form on the quotient manifold Q spanned by the coordinates y^a. We shall not assume that g_{ab} is necessarily positive definite everywhere on Q. The metric will be of Lorentzian signature if

$$g_{tt} g_{\psi\psi} - g_{t\psi}^2 < 0\,, \tag{2.2}$$

and g_{ab} is positive definite. The boundary of the region for which (2.2) holds, *i.e.* the null hypersurface $H \in M$ on which

$$g_{tt}\, g_{\psi\psi} - g_{t\psi}^2 = 0\,, \tag{2.3}$$

is a *Killing Horizon*, that is, a stationary null surface, invariant under time translations. Thus it can be traversed by timelike curves in only one direction. The null generator is

$$l = \partial_t + \Omega \partial_\psi\,, \tag{2.4}$$

where

$$\Omega = -\frac{g_{t\psi}}{g_{\psi\psi}}\Big|_H \tag{2.5}$$

The region $D \subset Q$ outside the horizon with boundary $\partial D = H$, is called in black holes physics the *domain of outer communication.*

We shall assume that the domain of outer communication contains a region for which $g_{tt} < 0$, so that ∂_t is timelike and the metric is stationary in this region. The *ergo-region* consists of points in Q for which $g_{tt} > 0$. The boundary ∂D is called the *ergo-sphere*, although topologically there is no reason why it should have the topology of a sphere. It is a general result that if the metric admits a Killing spinor, then the associated Killing vector ∂_t must always be non-spacelike, $g_{tt} \leq 0$, and and hence ergo-regions cannot exist. This also means that if one has a Killing horizon in this case, and if $g_{\psi\psi} > 0$, then we must have $g_{t\psi} = 0$ on the horizon, which implies that $\Omega = 0$, *i.e.* the horizon is non-rotating. This was first seen in the case of the BMPV black holes [28].

If $g_{\psi\psi} > 0$, and t is well-defined and non-periodic in some region, there cannot be closed timelike curves in that region, because we can use t as a *time function* such that t is non-decreasing along any future-directed timelike curve. In fact

$$g^{\mu\nu} \partial_\mu t\, \partial_\nu t = -\frac{g_{\psi\psi}}{g_{t\psi}^2 - g_{tt} g_{\psi\psi}}\,. \tag{2.6}$$

Thus the level sets $t =$ constant will be spacelike hypersurfaces, and traversable only once, as long as we are outside both the the horizon and $g_{\psi\psi} > 0$. Note that the coordinate function t certainly becomes singular on the Killing horizon, because the right-hand side of (2.6) becomes infinite there.

Conversely, if the condition $g_{\psi\psi} > 0$ is violated, then the metric will certainly admit closed timelike curves, or CTC's, in the region $C \subset Q$ for which $g_{\psi\psi} < 0$. Its boundary ∂C, given by $g_{\psi\psi} = 0$, will admit a closed null curve, or CNC. We shall refer to C as the *time machine* and its boundary as the *velocity of light surface* or VLS. As long as $g_{t\psi} \neq 0$ when $g_{\psi\psi} = 0$, the metric will remain non-singular there. Moreover the velocity of light surface will be a *timelike hypersurface*, and so timelike curves may cross into the time machine, and emerge from it, possibly earlier than when they entered. Because it is a timelike surface, the VLS is a distinct concept from that of a Cauchy horizon, which is necessarily a null hypersurface.

There are three cases of interest.

- The VLS is inside the horizon
- The VLS is on the horizon
- The VLS is outside the horizon

Of special interest is the last case. The metric may be re-expressed as

$$ds^2 =(g_{tt} - \frac{g_{t\psi}^2}{g_{\psi\psi}})dt^2 + g_{ab}dy^a dy^b + g_{\psi\psi}(d\psi + \omega + \frac{g_{t\psi}}{g_{\psi\psi}}dt)^2 \,. \tag{2.7}$$

The first two terms should now be a positive definite metric on the quotient $M/SO(2)$ of the manifold by the action of shifts in ψ. The orbits of ∂_ψ are timelike inside the time machine ($g_{\psi\psi} < 0$). The spacetime then comes to an end inside the time machine, where coefficient of dt^2 in the metric on the quotient

$$(g_{tt} - \frac{g_{t\psi}^2}{g_{\psi\psi}})dt^2 + g_{ab}dy^a dy^b \tag{2.8}$$

vanishes. We shall refer to this as the *pseudo-horizon.* In order that the spacetime be non-singular on the pseudo-horizon, it will in general be necessary to identify the coordinate t with an appropriate (real) period.

The time period is easily seen to be related to the formal expression for the surface gravity κ of the pseudo-horizon, namely

$$\kappa^2 = \nabla_\mu L \nabla^\mu L \,, \tag{2.9}$$

with

$$-L^2 = g_{tt} + 2g_{t\psi}\Omega + g_{\psi\psi}\Omega^2. \tag{2.10}$$

In fact one finds that κ^2 is negative. Formally, this suggests an imaginary temperature $T = \frac{2\pi}{\kappa}$.

In other words $\frac{2\pi}{|\kappa|}$ *is the period in real time, rather than the imaginary time period of the usual case.* For the same reason, the area of the horizon is purely imaginary in this case.

Of course it can be that the surface gravity κ actually vanishes on the horizon. Then, there is no need to make an identification of the time coordinate. This was found to occur in the case of the BMPV limiting solution. The resulting object has been referred to as a *repulson* [10]. An examination of geodesics in that example showed that they could not penetrate the horizon.

2.1.1 BMPV Black Hole

To see this in detail consider the metric of the supersymmetric BMPV black hole [9],

$$ds^2 = -\Delta^2(dt + \frac{am}{2\Delta r^2}\sigma_3)^2 + \frac{dr^2}{\Delta^2} + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)\,, \tag{2.11}$$

with a a constant and

$$\Delta = 1 - \frac{m}{r^2}\,. \tag{2.12}$$

The energy E and angular momentum J_R are

$$E = \frac{3\pi m}{4} \qquad J_R = -\frac{\pi a m}{2} \tag{2.13}$$

The horizon is located at $r = R_H = \sqrt{m}$ and the boundary of the time machine is located at $r = r_L = (am)^{\frac{1}{3}}$.

In the over-rotating case $a > m$, the boundary of the time machine lies outside the horizon. The surface gravity of the horizon vanishes and the area is

$$A = 2\pi^2\sqrt{r_H^6 - r_L^6}\,. \tag{2.14}$$

In this over-rotating case the area becomes imaginary because ∂_ψ is timelike.

If $\Delta_L = 1 - a^2m^2/r^6$, then the metric (2.11) may be written as

$$ds_5^2 = -\frac{\Delta^2\,dt^2}{\Delta_L} + \frac{dr^2}{\Delta^2} + \tfrac{1}{4}r^2\,\Delta_L\left(\sigma_3 - \frac{2am\Delta}{r^4\,\Delta_L}\,dt\right)^2 + \tfrac{1}{4}r^2\,(\sigma_1^2 + \sigma_2^2)\,. \tag{2.15}$$

In the over-rotating case, with Δ_L negative inside the time machine, the first two terms in (2.15) define a two-dimensional positive-definite metric. The horizon, $\Delta = 0$, is an infinite proper radial distance away. Defining a new radial coordinate $y = \Delta^{-1}$, the metric near the repulson at the horizon becomes approximately

$$ds_5^2 \sim \tfrac{1}{4}m\,\frac{dy^2}{y^2} + \frac{dt^2}{y^2\,|\Delta_L(r_H)|} + \tfrac{1}{4}m\,[\sigma_1^2 + \sigma_2^2 - |\Delta_L(r_H)|\,\sigma_3^2)\,, \tag{2.16}$$

where $\Delta_L(r_H) = 1 - a^2/m < 0$. The first two terms are the standard metric of negative curvature on the upper half plane $\mathfrak{Im}(z) > 0$, where

$$z = \frac{2t}{\sqrt{m\,|\Delta_L(r_H)|}} + \mathrm{i}\, y\,. \tag{2.17}$$

If one maps the upper half plane into the unit disc, then the repulson corresponds to a single point on the conformal boundary circle.

In the marginal case where Δ and Δ_L go to zero simultaneously, *i.e.* $r_H = r_L$, one obtains a singular object of zero area.

2.2 Frames rotating and non-rotating at infinity

When discussing the thermodynamics of rotating black holes in anti-de Sitter backgrounds, it is convenient to define energies and angular momenta with respect to a frame that is non-rotating at infinity. If, however, the system is supersymmetric, there is another natural frame, determined by the Killing vector associated with the Killing spinor. Consider, to begin with, the case of no black hole; *i.e.* pure AdS_n satisfying $R_{\mu\nu} = -(n-1)\, g^2\, g_{\mu\nu}$. There is a static Killing vector field $K = \partial/\partial t$, where t is the usual static time coordinate of AdS_n. However, the Killing spinors give rise to everywhere non-spacelike Killing vectors

$$K_\pm = \frac{\partial}{\partial t} + g \sum_i \epsilon_i\, \frac{\partial}{\partial \phi_i}\,, \tag{2.18}$$

where ϕ_i are the canonical angular coordinates in $[(n-2)/2]$ orthogonal spatial 2-planes, whose periods are 2π, and the signs of the $\epsilon_i = \pm 1$ the summation may be chosen independently for each i.

If one were to adapt the AdS_5 metric to a Killing vector K_+ with a spinorial square root, one would obtain a stationary metric. If one projects the metric orthogonal to the Killing field K_+, *i.e.* if one takes the quotient with respect to the one-parameter subgroup of isometries generated by K_+, one would obtain the (Einstein-Kähler) Bargmann metric on $SU(2,1)/U(2)$. In the pseudo-orthonormal frame adapted to this rotating frame, the Killing spinors are independent of time. In a pseudo-orthonormal frame adapted to a static Killing field, the spinors depend on time in a periodic fashion, being proportional to $\exp(i\frac{g}{2}t)$. Similarly they will have an angular dependence proportional to $\exp(\pm \mathrm{i}\frac{\psi}{2})$. Analogous remarks apply in other spacetime dimensions.

In general, there will always be a Killing vector asymptotic to K, but if black holes are present, it may well have an ergo-region in which $g(K,K) \equiv g_{\mu\nu}\, K^\mu K^\nu$ becomes positive. There will also always be Killing vectors $K_\pm$, and again in general there is no reason

why $g(K_\pm, K_\pm)$ should be everywhere non-positive. However, if the solution preserves $\mathcal{N}$ supersymmetries, *i.e.* it has $\mathcal{N}$ linearly-independent Killing spinors, for which

$$\bar{\epsilon}\,\gamma^\mu\,\epsilon\,\partial_\mu = K_\pm\,, \tag{2.19}$$

for some choices of the $\pm$ signs in (2.18). We shall refer to Killing vectors that are associated with Killing spinors in this way as having a spinorial square root. The spinorial square root need not be unique. In general the sum of two such Killing vectors will not have a spinorial square root. Note that even in a non-extremal solution, we can define $K_\pm$ as the Killing vectors that can be expressed asymptotically in terms of spinorial square roots, since any asymptotically AdS spacetime admits Killing spinors in the asymptotic region near infinity.

Consider one of these special Killing vector fields. It is future-directed, rotating "at the speed of light" at infinity, and is nowhere spacelike. It coincides, on the Killing horizon, with the null generator, and hence, for any supersymmetric black hole, the angular velocities on the horizon are $\pm g$. In other words, a supersymmetric black hole in an AdS background rotates with angular velocity $\pm g$ with respect to a frame that is non-rotating at infinity.

2.3 Geodesics

In order to study geodesic completeness, we need to be able to solve for the geodesics. All of the metrics considered in this paper take the form (2.1), with

$$g_{ab}dy^a dy^b = g_{rr}dr^2 + A^2(r)g_{ij}dx^i dx^j\,, \tag{2.20}$$

where

$$\omega = \omega_i(x)dx^i\,, \tag{2.21}$$

with $g_{tt}, g_{t\psi}$ and $g_{\psi\psi}$ being functions only of the coordinate r.

The Hamilton-Jacobi equation for neutral particles is

$$\begin{aligned}&\frac{1}{A^2(r)}g^{ij}(\partial_i S - \omega_i\partial_\psi S)(\partial_j S - \omega_j\partial_\psi S) + g^{rr}\partial_r S\partial_r S + g^{tt}\partial_t S\partial_t S\\ &+2g^{t\psi}\partial_t S\partial_\psi S + g^{\psi\psi}\partial_\psi S\partial_\psi S = -m^2.\end{aligned} \tag{2.22}$$

We separate the equation by setting

$$S = -Et + j\psi + W(r) + F(x^i)\,, \tag{2.23}$$

and find that

$$\frac{K^2}{A^2(r)} + g^{rr}(\frac{dW}{dr})^2 + E^2 g^{tt} - 2Ejg^{t\psi} + j^2 g^{\psi\psi} = -m^2, \tag{2.24}$$

with the constant K satisfying

$$g^{ik}(\partial_i F - j\omega_i)(\partial_k F - j\omega_k) = K^2\,. \tag{2.25}$$

The radial equation is then given by

$$\begin{aligned} m\frac{dr}{d\lambda} &= g^{rr}\partial_r W \\ &= \pm\Big(\frac{g^{rr}}{g_{t\psi}^2 - g_{tt}g_{\psi\psi}}\Big)^{\frac{1}{2}}\Big(E^2 g_{\psi\psi} + 2Ejg_{t\psi} + j^2 g_{tt} - (g_{t\psi}^2 - g_{tt}g_{\psi\psi})(m^2 + \frac{K^2}{A^2(r)})\Big)^{\frac{1}{2}}\,. \end{aligned} \tag{2.26}$$

The equation for x^i is determined in terms of the motion of a fictitious charged particle in a magnetic field ω_i moving on the manifold whose coordinates are x^i:

$$m\frac{dx^i}{d\lambda} = \frac{g^{ik}}{A^2(r)}(\partial_k F - j\omega_k)\,. \tag{2.27}$$

The motion in time and angle is given by

$$\begin{aligned} m\frac{dt}{d\lambda} &= \frac{1}{g_{t\psi}^2 - g_{tt}g_{\psi\psi}}(Eg_{\psi\psi} + jg_{t\psi})\,, \\ m\frac{d\psi}{d\lambda} &= \frac{1}{g_{t\psi}^2 - g_{tt}g_{\psi\psi}}(-jg_{tt} - Eg_{t\psi})\,. \end{aligned} \tag{2.28}$$

If the time machine is outside the horizon, then from (2.26) it follows that particles with $j = 0$ cannot enter the time machine. However, by taking E and j such that $2g_{t\psi}Ej$ sufficiently large and positive, one can find geodesics where $dr/d\lambda$ is still negative as r approaches r_L, and so such geodesics can enter the time machine.

For particles with charge e moving in a potential A_μ, the the geodesics are governed by the Hamilton-Jacobi equation with the replacement

$$\partial_\mu S \to \partial_\mu S + eA_\mu\,, \tag{2.29}$$

so that

$$m\frac{dx^\mu}{d\lambda} = g^{\mu\nu}(\partial_\nu S + eA_\nu))\,. \tag{2.30}$$

If we assume that

$$A = \phi(r)dt + \chi(r)(d\psi + \omega_i dx^i) \tag{2.31}$$

the discussion above will go through as before with the replacements

$$E \to E - e\phi(r)\,, \qquad j \to j + e\chi(r)\,. \tag{2.32}$$

One may think of ϕ an electrostatic potential and χ as a magnetostatic potential.

2.4 Energetic Considerations

From (2.26), it follows that a neutral particle may only cross the horizon if

$$E^2 g_{\psi\psi} + 2Ej\, g_{t\psi} + j^2 g_{tt} \geq 0 \tag{2.33}$$

on the horizon. Bearing in mind that the quadratic form

$$\Omega^2 g_{\psi\psi} + 2\Omega g_{t\psi} + g_{tt} \tag{2.34}$$

has a pair of coincident roots $\Omega = -g_{t\psi}/g_{\psi\psi}$ on the horizon, we must have

$$g_{\psi\psi}\, (E - \Omega j)^2 > 0\,. \tag{2.35}$$

If the time machine is inside the horizon then $g_{\psi\psi} > 0$, and hence, assuming $E > 0$, we must have

$$E - \Omega\, j > 0\,. \tag{2.36}$$

Since E and j give the small increment in the energy and angular momentum of the black hole, one may regard this inequality as a statement of the second law of thermodynamics. If, on the other hand, the time machine is outside the horizon, then $g_{\psi\psi} < 0$ on the horizon, and (2.35) can never be satisfied. This means that the repulson is a barrier preventing penetration by timelike geodesics.

In the case of charged particles, one defines the electrostatic potential of the horizon by $\Phi = l^\mu\, A_\mu$, where $l^\mu \partial_\mu = \partial_t + \Omega \partial_\psi$ is the null generator of the horizon. This implies

$$\Phi = (\phi + \Omega \chi)\Big|_H\,. \tag{2.37}$$

The second law for the case of a time machine inside the horizon then generalises from (2.36) to become

$$E - \Omega j - \Phi e > 0\,. \tag{2.38}$$

2.5 Quantisation Conditions

The vector potential (2.31) in general becomes singular at the horizon, as can be seen from its norm

$$A_\mu\, A_\nu\, g^{\mu\nu} = \frac{\chi^2\, g_{tt} + \phi^2\, g_{\psi\psi} - 2\phi\chi\, g_{t\psi}}{g_{tt}\, g_{\psi\psi} - g_{t\psi}^2}\,. \tag{2.39}$$

It is then necessary to make a gauge transformation

$$A \to \widetilde{A} = A - d(\phi(r_+)\, t)\,. \tag{2.40}$$

A field Ψ with charge e will thus suffer a gauge transformation

$$\Psi \rightarrow \exp(\mathrm{i}\, e\, \phi(r_+)\, t)\, \Psi\,. \tag{2.41}$$

If the coordinates t is periodic, with period Δt, this can lead to the quantisation condition

$$\frac{e\, \phi(r_H)\, \Delta t}{2\pi} \in \mathbb{Z}\,. \tag{2.42}$$

If the original gauge potential A and the new potential $\widetilde{A}$ are both needed in order to define the connection over the entire space, then the $U(1)$ transition function given by (2.41) will be well-defined only if it is a periodic function of t such that (2.42) is satisfied. Typically, the need for the two gauge patches arises if the potential A falls off sufficiently rapdidly at infinity but $\widetilde{A}$ does not.

The quantisation condition (2.42) is unfamiliar since one does not normally consider time to be periodic. An analogous condition arises when considering the time-dependent Josephson effect, and so we propose calling it the Josephson quantisation condition.

3 Rotating Black Holes and Supersymmetric Limits in Five-Dimensional Gauged Supergravity

3.1 Black-Hole Thermodynamics in Five Dimensions

Our starting point is the non-extremal charged rotating black holes in five-dimensional gauged supergravity, which were found in [16, 17]. Specifically, we shall focus on the solutions obtained in [17], which describe rotating black holes with the two rotation parameters set equal, and with three independent electric charges carried by the three commuting $U(1)$ gauge fields in the maximal $SO(6)$ gauged theory. The relevant part of the supergravity Lagrangian is given by

$$e^{-1}\, \mathcal{L} = R - \tfrac{1}{2}\partial\vec{\varphi}^2 - \tfrac{1}{4}\sum_{i=1}^{3} X_i^{-2}\, \big(F^i\big)^2 + 4g^2\, \sum_{i=1}^{3} X_i^{-1} + \tfrac{1}{24}|\epsilon_{ijk}|\, \epsilon^{\mu\nu\rho\sigma\lambda} F^i_{\mu\nu}\, F^j_{\rho\sigma}\, A^k_\lambda\,, \tag{3.1}$$

where $\vec{\varphi} = (\varphi_1, \varphi_2)$, and

$$X_1 = e^{-\frac{1}{\sqrt{6}}\varphi_1 - \frac{1}{\sqrt{2}}\varphi_2}\,, \qquad X_2 = e^{-\frac{1}{\sqrt{6}}\varphi_1 + \frac{1}{\sqrt{2}}\varphi_2}\,, \qquad X_3 = e^{\frac{2}{\sqrt{6}}\varphi_1}\,. \tag{3.2}$$

The solutions in [17] are characterised by five non-trivial parameters, associated with the mass, the angular momentum, and the three electric charges. As presented in [17], the metrics were written with an additional trivial parameter called γ, which characterises the

asymptotic rotation rate as measured from infinity. For convenience, we shall set $\gamma = 0$, which means that the metric is asymptotically non-rotating. The solution of [17] is then given by

$$
\begin{aligned}
ds_5^2 &= (H_1 H_2 H_3)^{1/3} \Big\{ -\frac{r^2\, Y}{f_1}\, dt^2 + \frac{r^4}{Y}\, dr^2 + \frac{f_1}{4r^4\, H_1 H_2 H_3}\, (\sigma_3 - \frac{2 f_2}{f_1}\, dt)^2 \\
&\qquad + \tfrac14 r^2 (\sigma_1^2 + \sigma_2^2) \Big\}\,, \\
A^i &= \frac{2m}{r^2\, H_i} \Big(s_i\, c_i\, dt + \tfrac12 a\, (c_i\, s_j\, s_k - s_i\, c_j\, c_k)\, \sigma_3 \Big)\,, \\
X_i &= \frac{R}{r^2\, H_i}\,, \qquad i = 1,2,3\,,
\end{aligned}
\tag{3.3}
$$

where

$$
\begin{aligned}
H_i &= 1 + \frac{2m\, s_i^2}{r^2}\,, \\
\sigma_1 + \mathrm{i}\, \sigma_2 &= e^{-\mathrm{i}\, \psi}\, (d\theta + \mathrm{i}\, \sin\theta\, d\varphi)\,, \qquad \sigma_3 = d\psi + \cos\theta\, d\varphi\,,
\end{aligned}
\tag{3.4}
$$

and s_i and c_i are shorthand notations for

$$
s_i \equiv \sinh \delta_i\,, \qquad c_i \equiv \cosh \delta_i\,, \qquad i = 1,2,3\,. \tag{3.5}
$$

Note that in the expressions in (3.3) for the vector potentials A^i, the triplet indices (i, j, k) are all unequal: $(i \ne j \ne k \ne i)$. The functions f_1, f_2, f_3 and Y are given by

$$
\begin{aligned}
f_1 &= r^6\, H_1 H_2 H_3 + 2m a^2 r^2 + 4m^2 a^2 [2(c_1 c_2 c_3 - s_1 s_2 s_3) s_1 s_2 s_3 - s_1^2 s_2^2 - s_2^2 s_3^2 - s_3^2 s_1^2]\,, \\
f_2 &= 2ma(c_1 c_2 c_3 - s_1 s_2 s_3) r^2 + 4m^2 a s_1 s_2 s_3\,, \\
f_3 &= 2ma^2(1 + g^2 r^2) + 4g^2 m^2 a^2 [2(c_1 c_2 c_3 - s_1 s_2 s_3) s_1 s_2 s_3 - s_1^2 s_2^2 - s_2^2 s_3^2 - s_3^2 s_1^2]\,, \\
Y &= f_3 + g^2 r^6 H_1 H_2 H_3 + r^4 - 2m r^2\,.
\end{aligned}
\tag{3.6}
$$

(Here we have renamed the parameters μ and ℓ in [17] as $2m$ and a respectively.)

We wish to calculate the conserved charges corresponding to the energy E, the angular momentum J and the three electric charges Q_i. The electric charges are easily calculated from

$$
Q_i = \frac{1}{8\pi} \int_{S_3} (X_i^{-2} {*F^i} - \tfrac12 |\epsilon_{ijk}|\, A^j \wedge F^k)\,, \tag{3.7}
$$

where the integration is performed over the 3-sphere at infinity.[1] The angular momentum can be derived from the Komar integral

$$
J = \frac{1}{16\pi} \int_{S^3} {*dK}\,, \tag{3.8}
$$

[1]The prefactor $1/(8\pi)$ rather than $1/(4\pi)$ results from our normalisation $\mathcal{L} \sim R - \frac14 F^2$ rather than $R - F^2$ in the Lagrangian.

where $K = K_\mu\, dx^\mu$, and $K^\mu\, \partial_\mu = \partial/\partial\psi$ is the rotational Killing vector conjugate to the angular momentum. The calculation of the energy E is trickier because the analogous Komar integral for the relevant timelike Killing vector diverges in an asymptotically AdS background, and requires a delicate and somewhat ambiguous regularisation. As discussed in [29], one of the simplest ways of calculating the energy in such a situation is to integrate up the first law of thermodynamics, which in this case reads[2]

$$dE = T\, dS + 2\Omega\, dJ + \sum_i \Phi_i\, dQ_i\,, \tag{3.9}$$

where T is the Hawking temperature, S is the area of the horizon, Ω is the angular velocity of the horizon relative to the frame that is non-rotating at infinity, and Φ_i are the electrostatic potentials on the horizon. Since all the other quantities are relatively easily calculated without ambiguity, this provides a convenient way to compute the energy.[3]

For the rotating black holes (3.3), we have

$$\begin{aligned} S &= \tfrac12 \pi^2\, \sqrt{f_1}, \qquad T = \frac{Y'}{4\pi\, r\, \sqrt{f_1}}\,, \qquad \Omega = \frac{2f_2}{f_1}\,,\\ \Phi_i &= \frac{2m}{r^2\, H_i}\, \Big(s_i c_i - \tfrac12 a\, \Omega\, (c_i s_j s_k - s_i c_j c_k)\Big)\,, \end{aligned} \tag{3.10}$$

with all quantities evaluated on the outer horizon $r = r_+$ where $Y(r)$ has its largest root. Note that Ω here is the angular velocity measured with respect to the Killing vector $\partial/\partial\psi$. In terms of azimuthal coordinates ϕ_1 and ϕ_2, with canonical periods 2π, in a generic solution with independent angular velocities Ω_1 and Ω_2 in the two orthogonal 2-planes, one has $\psi = \phi_1 + \phi_2$, $\varphi = \phi_1 - \phi_2$, and hence $\partial/\partial\psi = \frac12 \partial/\partial\phi_1 + \frac12 \partial/\partial\phi_2$, and hence $\Omega = \frac12(\Omega_1 + \Omega_2)$. When the two angular velocities are equal, as in our case, one therefore has $\Omega = \Omega_1 = \Omega_2$.

After calculations of some complexity, we can integrate the first law (3.9) to obtain the energy E. Ours results for the various conserved quantities in this case are:

$$\begin{aligned} E &= \tfrac14 m\, \pi\, (3 + a^2 g^2 + 2s_1^2 + 2s_2^2 + 2s_3^2)\,,\\ J &= \tfrac12 m\, a\, \pi\, (c_1 c_2 c_3 - s_1 s_2 s_3)\,,\\ Q_i &= \tfrac12 m\, \pi\, s_i c_i\,. \end{aligned} \tag{3.11}$$

[2]The coefficient 2 in the angular momentum contribution comes from $\Omega_1\, dJ_1 + \Omega_2\, dJ_2$, with the two angular momentum contributions being equal.

[3]There are unambiguous methods available for directly computing the energy of an asymptotically AdS spacetime, such as the conformal definition of the conserved energy given by Ashtekar, Magnon and Das (AMD) [30, 31]. In fact in [29] the energies of rotating AdS black holes in all dimensions were calculated both from the integration of the first law, and by the AMD procedure, and the two results were shown to be identical.

It should be emphasised that the fact one obtains an exact form on the right-hand side of (3.9), and hence that it can be integrated, is a somewhat non-trivial result, which, in its own right, provides a significant test of the validity of the first law for these black hole solutions.

If one turns off the charges, by setting $s_i = 0$, these expressions reduce to ones obtained for five-dimensional rotating AdS black holes in [29]. If the three charges are set equal, $s_i = s$, the expressions reduce to the ones found in [32].

3.2 Supersymmetric Limits and the Supersymmetric Bound

The algebra of the supercharges $\mathcal{Q}$ in five-dimensional $\mathcal{N} = 2$ gauged supergravity, *i.e.* the $U(1)$ gauged minimal supergravity, is given by $\{\mathcal{Q}, \mathcal{Q}\} = \{\overline{\mathcal{Q}}, \overline{\mathcal{Q}}\} = 0$ and

$$M_{\equiv}\{\mathcal{Q}, \overline{\mathcal{Q}}\} = \tfrac{1}{2}\, J_{AB}\, \gamma^{AB} + Z\,, \tag{3.12}$$

where γ^{AB} are generators of $SO(4,2)$ in the 8×8 spin representation, and the supercharge Q is a Weyl spinor of $SO(4,2)$. J_{05} is the energy, and J_{ij} for $1 \le i \le 4$ correspond to angular momenta.[4] We are interested in the case where there are two rotation parameters J_1 and J_2, describing rotations in the J_{12} and J_{34} planes. We have

$$J_{05} = g^2\, E\,, \qquad J_{12} = g^3\, J_1\,, \qquad J_{34} = g^3\, J_2\,, \qquad Z = g^2 \sum_i Q_i\,. \tag{3.13}$$

The Bogomolny matrix $g^{-2}\, M$ has in general four distinct eigenvalues associated with four complex Weyl eigenspinors of the given chirality. These are given by

$$\begin{aligned}
\lambda &= E + gJ_1 + gJ_2 - \sum_i Q_i\,,\\
\lambda &= E - gJ_1 - gJ_2 - \sum_i Q_i\,,\\
\lambda &= E + gJ_1 - gJ_2 + \sum_i Q_i\,,\\
\lambda &= E - gJ_1 + gJ_2 + \sum_i Q_i\,,
\end{aligned} \tag{3.14}$$

each with degeneracy 1.

We could instead have used a notation where the supercharges were Majorana. This would lead to each of the four eigenvalues in (3.14) being associated with two real spinors.

[4] The corresponding four-dimensional boundary theory is usually referred to as an $\mathcal{N} = 1$ superconformal theory.

If the two angular momenta are set equal, $J_1 = J_2 = J$, we obtain

$$\lambda = E + 2g\,J - \sum_i Q_i \qquad \text{once}\,, \tag{3.15}$$

$$\lambda = E - 2g\,J - \sum_i Q_i \qquad \text{once}\,, \tag{3.16}$$

$$\lambda = E + \sum_i Q_i \qquad \text{twice}\,. \tag{3.17}$$

The vanishing of one of these eigenvalues is associated with the occurrence of supersymmetries (see [33, 34]). We shall refer to the situation where states have vanishing eigenvalues given by (3.17) as supersymmetric configurations of type A. By contrast, the situation where the eigenvalue in (3.15) or in (3.16) vanishes will be associated with supersymmetric configurations of type B. Note that the the identification of the R-charges is confirmed by both the local supersymmetry transformation rules and the Witten-Nester identity [33].

The discussion above is given in terms of the $\mathcal{N} = 2$ theory, which can be embedded in the maximal $\mathcal{N} = 8$ theory, for which there are four complex Weyl supercharges and the R-symmetry is $SU(4)$. The Bogomolny matrix now carries R-symmetry indices, and in general has 16 distinct eigenvalues for Weyl eigenspinors of the given chirality. These eigenavlues are

$$\lambda = E \pm g j_1 \pm g J_2 \pm Q_1 \pm Q_2 \pm Q_3\,, \tag{3.18}$$

where, according to the convention chosen for the chirality, the total number of minus signs must be either always odd, or always even. Again, one could instead use a notation where the supercharges were Majorana. In this case each of the 16 eigenvalues would be associated with two Majorana eigenspinors.

Taking our results (3.11) for the conserved charges of the non-extremal rotating AdS black holes in five dimensions, we find that

$$\begin{aligned}
E + 2g\,J - \sum_i Q_i &= \frac{m\pi}{4}(1 + ag\,e^{\delta_1+\delta_2+\delta_3})(e^{-2\delta_1} + e^{-2\delta_2} + e^{-2\delta_3} + ag\,e^{-\delta_1-\delta_2-\delta_3})\,,\\
E - 2g\,J - \sum_i Q_i &= \frac{m\pi}{4}(1 - ag\,e^{\delta_1+\delta_2+\delta_3})(e^{-2\delta_1} + e^{-2\delta_2} + e^{-2\delta_3} - ag\,e^{-\delta_1-\delta_2-\delta_3})\,,\\
E + \sum_i Q_i &= \frac{m\pi}{4}(e^{2\delta_1} + e^{2\delta_2} + e^{2\delta_3} + a^2 g^2)\,.
\end{aligned} \tag{3.19}$$

We see that vanishing of these quantities can be achieved in the following cases:

$$E + 2g\,J - \sum_i Q_i = 0 \qquad \text{if} \quad e^{\delta_1+\delta_2+\delta_3} = -\frac{1}{ag}\,, \tag{3.20}$$

$$E - 2g\,J - \sum_i Q_i = 0 \qquad \text{if} \quad e^{\delta_1+\delta_2+\delta_3} = \frac{1}{ag}\,, \tag{3.21}$$

$$E+\sum_i Q_i \quad = \quad 0 \qquad \text{if} \quad m\to 0\,, \quad m\,e^{-2\delta_i}=2q_i\,, \tag{3.22}$$

where in the last case the limit is achieved by sending the δ_i parameters to $-\infty$ whilst keeping the q_i fixed.[5]

The positivity of the left-hand side of the supersymmetry algebra (3.12) implies quite generally that the AdS and R-charges live in a convex cone invariant under the adjoint action of the product of the R-symmetry group and the anti-de-Sitter group. This invariant cone may be rotated to lie in the maximal torus, which is spanned by the energy, angular momenta and U(1) charges. The boundary of the cone consists of states with some degree of supersymmetry. The boundary is a stratified set consisting of faces, edges, etc., with the strata of smaller dimension consisting of increasingly larger numbers of zero eigenvalues, *i.e.* of increasing amounts of supersymmetry [35].

In this $D=5$ example, truncated to a single R-charge $\sum_i Qi$, supersymmetry allows only states labelled by points in the four dimensional vector space with coordinates E, gJ_1, gJ_2 and $\sum_i Qi$, which satisfy the four inequalities

$$\begin{aligned}
E+gJ_1+gJ_2-\sum_i Q_i &\geq 0\,,\\
E-gJ_1-gJ_2-\sum_i Q_i &\geq 0\,,\\
E+gJ_1-gJ_2+\sum_i Q_i &\geq 0\,,\\
E-gJ_1+gJ_2+\sum_i Q_i &\geq 0\,.
\end{aligned}\tag{3.23}$$

This is a convex cone in $\mathbb{R}^4$, with four faces given by the four hyperplanes through the origin, whose equations are given by the saturation of the inequalities above, *i.e.* when one of the four eigenvalues vanishes. The base of the cone is obtained by setting $E=$ constant >0 in these inequalities, and is easily seen to consist of points in $\mathbb{R}^3$, inside or on a regular tetrahedron. This has four faces, corresponding to the vanishing of one of the four eigenvalues. If $J_1=J_2=J$, then the third and fourth faces intersect on an edge of

[5] The other ostensibly supersymmetric cases that one might think could arise, such as if $(e^{-2\delta_1}+e^{-2\delta_2}+e^{-2\delta_3}+ag\,e^{-\delta_1-\delta_2-\delta_3})=0$ in the first line of (3.19), are in fact spurious, and are not associated with supersymmetric limits of the black holes we are discussing. This can be verified by checking the explicit supergravity transformation rules, or, more easily, by checking to see whether either of the "spinorial square root" Killing vectors K_+ or K_- introduced in section 2.2 has a manifestly non-positive norm, as would have to be the case if there existed a Killing spinor in the background. Evidently the solutions corresponding to one of these spurious roots have singularities that violate the assumptions under which one can deduce supersymmetric from a vanishing eigenvalue of the Bogomolny matrix.

enhanced supersymmetry for which

$$E+\sum_i Qi=0\,. \tag{3.24}$$

Thus states on the faces are what was earlier called type B. The states on the edges are what was earlier called type A.

3.3 $E+\sum Q_i=0$: Case A

This solution preserves $\frac{1}{2}$ of the $\mathcal{N}=2$ supersymmetry, since there are two zero eigenvalues of the Bogomolny matrix in this case (see equation (3.17)). The supersymmetric condition $E+\sum Q_i=0$ can be satisfied by taking $m\to 0$ and $\delta_i\to-\infty$, whilst leaving $q_i=\frac{1}{2}m\,e^{-2\delta_i}$ fixed. If It is also necessary to scale the parameter a according to

$$a=\tfrac{1}{2}\alpha\sqrt{\frac{2m}{q_1q_1q_3}}\,, \tag{3.25}$$

so that the gauge fields remains finite. After taking $m\to 0$, the solution then becomes

$$\begin{aligned}
ds^2 &= -\tfrac{1}{4}r^2\,\frac{V}{B}\,dt^2+\frac{dr^2}{V}+B\,(\sigma_3+f\,dt)^2+\tfrac{1}{4}R^2(\sigma_1^2+\sigma_2^2)\,,\\
A^i &= -\frac{1}{r^2\,H_i}(q_i\,dt-\tfrac{1}{2}\alpha\sigma_3)\,,
\end{aligned} \tag{3.26}$$

where

$$\begin{aligned}
&V=\frac{r^4+g^2r^6\,H_1H_2H_3-g^2\alpha^2}{r^4\,(H_1H_2H_3)^{1/3}}\,,\qquad B=\frac{r^6H_1H_2H_3-\alpha^2}{4r^4(H_1H_2H_3)^{2/3}}\,,\\
&f=-\frac{2\alpha\,r^2}{r^6H_1H_2H_3-\alpha^2}\,,\qquad R^2=r^2(H_1H_2H_3)^{1/3}\,,\qquad H_i=1+\frac{q_i}{r^2}\,.
\end{aligned} \tag{3.27}$$

From (3.11), the energy, angular momentum and the charges are given by

$$E=\tfrac{1}{4}\pi(q_1+q_2+q_3)\,,\qquad J=\tfrac{1}{4}\pi\,\alpha\,,\qquad Q_i=-\tfrac{1}{4}\pi\,q_i\,, \tag{3.28}$$

Note that we have

$$g(K,K)=-\frac{Vr^2}{4B}+Bf^2=-\frac{g^2r^2H_1H_2H_3+1}{(H_1H_2H_3)^{2/3}}\,, \tag{3.29}$$

where K is the asymptotically non-rotating timelike Killing vector introduced in section 2.2, and which is given by $K=\partial/\partial t$ in this case. The function $g(K,K)$ is negative definite provided $r^2H_i>0$. Thus the Killing vector K is nowhere spacelike, but since it is non-rotating at infinity it does not have a spinorial square root. It is straightforward to calculate

$g(K_+, K_+)$ for the Killing vector $K_+ = \partial/\partial t + 2g\,\partial/\partial\psi$ that does have a spinorial square root; we find

$$g(K_+, K_+) = -\frac{V r^2}{4B} + B(f+2g)^2 = -\frac{(r^2+g\alpha)^2}{r^4(H_1H_2H_3)^{2/3}}\,. \tag{3.30}$$

This is indeed manifestly non-positive.

The solutions (3.26) were obtained previously in [22]. Here we examine further their global properties. Note that the range of the coordinate r is $r^2+q\ge 0$, with $r_0^2=-q$ being the singularity, where $q=\min\{q_1,q_2,q_3\}$, with $q_i>0$. Thus there must exist a VLS where $B=0$ for non-vanishing rotation, inside which there are CTC's. If V is never zero for all $r^2+q\ge 0$, the solution has a naked singularity. This happens when there is no rotation (without CTC's), or under-rotation (with CTC's).

Since the right-hand side of the (3.29) is negative definite, it follows that if there is a Killing horizon where $V=0$, then in general B must be negative, implying CTC's. This in fact rules out any regular black hole solution. The only possibility for the solution not to have a naked CTC is when both B and V approach zero simultaneously.[6] This can occur when $\alpha^2=q_1q_2q_3$. We shall now discuss this case in detail.

3.3.1 Critical rotation: $\alpha^2 = q_1q_2q_3$

In this case, there is neither an horizon nor a time machine, and the radial coordinate r runs from 0 to ∞. Near to $r=0$, the metric becomes

$$\begin{aligned} ds^2 \;\sim\;& -g^2(q_1q_2q_3)^{1/3}\,dt^2 + \frac{q_1q_2+q_1q_3+q_2q_3}{4(q_1q_2q_3)^{2/3}}\,r^2\Big(\sigma_3 - \frac{2(q_1q_2q_3)^{1/2}}{q_1q_2+q_1q_3+q_2q_3}dt\Big)^2\\ &+\frac{(q_1q_2q_3)^{1/3}\,dr^2}{g^2(q_1q_2+q_1q_3+q_2q_3)} + \tfrac14(q_1q_2q_3)^{1/3}(\sigma_1^2+\sigma_2^2)\,, \end{aligned} \tag{3.31}$$

A conical singularity at $r=0$ is avoided if the charges are chosen to satisfy the condition

$$\frac{g(q_1q_2+q_1q_3+q_2q_3)}{k\,\sqrt{q_1q_2q_3}} = 1\,,\qquad k=\frac{g\,\alpha}{k}\sum_i\frac{1}{q_i}\in\mathbb{Z}_+ \tag{3.32}$$

with ψ having period $\Delta\psi = 4\pi/k$ for any strictly positive integer k.

Setting $dt=0$ in the metric (3.26) with $\alpha^2=q_1q_2q_3$ gives a four-dimensional metric that is everywhere positive-definite, except at the coordinate singularity at $r=0$. From the behaviour near $r=0$ given in (3.31), we have seen that the metric may be extended to give a complete non-singular metric on an $\mathbb{R}^2$ bundle over S^2, with the coordinates r and

[6] In the identity (3.30), it appears that the right-hand side could be zero if we choose to have $\alpha=-r_0^2/g$. This implies that $H_1H_2H_3=0$ at $r=r_0$ for the existence of a horizon. This however forces B to be divergent at $r=r_0$.

ψ parameterising the $\mathbb{R}^2$, and θ and ϕ are coordinates on the S^2 base. Such $\mathbb{R}^2$ bundles are characterised topologically by the single integer k, which gives the Chern number of the bundle. If $k = 1$ we obtain the spin bundle of S^2, *i.e.* the Taub-BOLT manifold [36], and if $k = 2$ we obtain the tangent bundle of S^2, *i.e.* the Eguchi-Hanson manifold [37]. [7] The total five-dimensional spacetime is topologically a product of the four-dimensional manifold with the real line. It is a spin manifold if and only if the $\mathbb{R}^2$ bundle over S^2 is a spin manifold, and this is the case if and only if k is even. If k is odd, the manifolds admit a spin^c structure, *i.e.* one may consistently couple spinors provided they carry an appropriate $U(1)$ charge with respect to a suitable Maxwell gauge field [38]. A brief discussion of the spin^c structure of the Taub-BOLT metric is given in appendix A.

It is interesting to note that these topological solitons, which as we have seen have zero temperature, still satisfy the first law of thermodynamics, namely, we have

$$dE = 2\Omega\, dJ + \sum_i \Phi_i dQ_i \,. \tag{3.33}$$

We now turn to the regularity of the gauge fields. The gauge potentials in (3.26) are well-defined for all $r > 0$, with, in particular

$$A^i_\mu\, A^i_\nu\, g^{\mu\nu} \sim \frac{g^2\, q_1 q_2 q_3 - q_i^2}{g^2 r^6} \tag{3.34}$$

as $r \longrightarrow \infty$. They are, however, singular at $r = 0$, diverging as

$$A^i_\mu\, A^i_\nu\, g^{\mu\nu} \sim \frac{(q_1 q_2 q_3)^{5/3}}{q_i^2 (q_1 q_2 + q_2 q_3 + q_3 q_1)\, r^2} \tag{3.35}$$

at small r. We can define new gauge potentials, $A^{i\prime}$ and $A^{i\prime\prime}$ that are well-defined at $r = 0$, $\theta = 0$, and at $r = 0$, $\theta = \pi$ respectively, by means of the gauge transformations

$$\begin{aligned} A^i &\longrightarrow A^{i\prime} = A^i + \frac{\alpha}{2q_i}\,(d\psi + d\phi)\,, \\ A^i &\longrightarrow A^{i\prime\prime} = A^i + \frac{\alpha}{2q_i}\,(d\psi - d\phi)\,. \end{aligned} \tag{3.36}$$

These potentials are, however, not well-defined at infinity, since one has

$$A^{i\,\prime}_\mu\, A^{i\,\prime}_\nu\, g^{\mu\nu} \sim \frac{2\alpha^2\, g^2(1-\cos\theta)}{r^2\, \cos^2\theta}\,, \qquad A^{i\,\prime\prime}_\mu\, A^{i\,\prime\prime}_\nu\, g^{\mu\nu} \sim \frac{2\alpha^2\, g^2(1+\cos\theta)}{r^2\, \cos^2\theta} \tag{3.37}$$

as r approaches infinity. Thus we must necessarily cover the manifold in three gauge patches, and associated with these are quantisation conditions arising when we require the quantum

[7] Of course the *metrics* on these manifolds, given in [36] and [39], differ from those we are finding in this paper, but the *topologies* are the same. The correct topology for the Eguchi-Hanson metric was first discussed in [37].

well-definedness of the gauged supergravity's fermionic wave functions, which are all gauged with respect to the graviphoton $U(1)$ connection $\frac{1}{2}g\,(A^1_\mu + A^2_\mu + A^3_\mu)$. The wave functions in the three gauge patches are related by the transition functions

$$\begin{aligned} U_1 &= \exp(\tfrac{\mathrm{i}}{2}\,g\,\alpha\,\sum_i \frac{1}{q_i}\,\phi) = e^{\frac{1}{2}\mathrm{i}\,k\,\phi}\,,\\ U_2 &= \exp(\tfrac{\mathrm{i}}{4}\,g\,\alpha\,\sum_i \frac{1}{q_i}\,(\psi+\phi)) = e^{\frac{1}{4}\mathrm{i}\,k\,(\psi+\phi)}\,,\\ U_3 &= \exp(\tfrac{\mathrm{i}}{4}\,g\,\alpha\,\sum_i \frac{1}{q_i}\,(\psi-\phi)) = e^{\frac{1}{4}\mathrm{i}\,k\,(\psi-\phi)}\,. \end{aligned} \tag{3.38}$$

(The second equalities on each line follow from (3.32).)

If the spacetime manifold admits a spin structure, these transition functions must be appropriately single-valued. Thus U_1, which governs the behaviour of the graviphoton $U(1)$ bundle restricted to the S^2 base, must satisfy $U_1(\phi+2\pi) = U_1(\phi)$, whilst U_2 and U_3, which govern the behaviour of the graviphoton $U(1)$ bundle restricted to the $\mathbb{R}^2$ fibres, must satisfy $U_2(\psi+4\pi/k) = U_2(\psi)$ and $U_3(\psi+4\pi/k) = U_3(\psi)$. As we discussed earlier, the condition for having a spin structure is that k be even. Thus although U_1 is indeed then single-valued, we see that U_2 and U_3 are not, since ψ has period $4\pi/k$.

If, on the other hand, the spacetime is not a spin manifold, which happens if k is odd, then the fermions must be sections of a spin^c bundle, as we discussed above. This implies that the transition functions should satisfy $U_1(\phi+2\pi) = -U_1(\phi)$, and $U_2(\psi+4\pi/k) = -U_2(\psi+4\pi/k)$, $U_3(\psi+4\pi/k) = -U_3(\psi+4\pi/k)$. The minus signs in these relations precisely cancel the minus signs coming from the transition functions in the spin connection that are responsible for the absence of an ordinary spin structure [38].

The argument given above, based on the transition functions for the graviphoton $U(1)$ bundle over the spatial manifold, itself an $\mathbb{R}^2$ bundle over S^2, with connection $\frac{1}{2}g\,\sum_i A^i$, may be expressed more concisely checking the Dirac quantisation condition, which for a spin structure is

$$\frac{g}{4\pi}\int_C \sum_i F^i \in \mathbb{Z}\,, \tag{3.39}$$

and for a spin^c structure is

$$\frac{g}{4\pi}\int_C \sum_i F^i \in \mathbb{Z} + \tfrac{1}{2}\,, \tag{3.40}$$

where the integrals are taken over all relevant 2-cycles C. Because the spatial manifold is non-compact, the cycles in our case consist of the S^2 base at $r=0$, and the non-compact

cycle corresponding to the $\mathbb{R}^2$ fibre over any point of the S^2 base. These give

$$\frac{g}{4\pi}\int_{S^2}\sum_i F^i = -\tfrac{1}{2}k\,, \qquad \frac{g}{4\pi}\int_{\mathbb{R}^2}\sum_i F^i = -\tfrac{1}{2}\,. \tag{3.41}$$

When k is even, in which case the spacetime is a spin manifold, we see that first integral indeed satisfies the spin condition (3.39), but the second integral can never satisfy this condition. When k is odd, in which case the spacetime is a spin^c manifold, we see that both the integrals in (3.41) satisfy the spin^c conditions (3.40). Thus we conclude, in agreement with our discussion of the transition functions above, that the fermions of the supergravity multiplet are quantum-mechanically well-defined in the case that k is odd, but not when k is even. It should be emphasised, however, that in all cases the bosonic soliton solutions are completely regular.

3.3.2 General rotation: $\alpha^2 \neq q_1 q_2 q_3$

In this case, if the parameters α, q_1, q_2 and q_3 lie in appropriate ranges, there is a Killing horizon at $r^2 = r_+^2$, the largest value of r^2 where V vanishes, and at which $B < 0$. The functions metric V and B depend on r only via r^2, and so the values of r^2 at which they vanish might in fact be negative. The metric (3.26) itself is real and of Lorentzian signature for values of r^2, including negative ones, such that $r^2 + q_i > 0$ for all i. With this understanding, the VLS occurs at $r^2 = r_L^2 > r_+^2$, where $B(r_L) = 0$. In the region between, $r_+^2 \le r^2 < r_L^2$, the metric has CTC's. Since the Killing horizon at $r = r_+$ lies inside the VLS, it is what we have defined earlier as a pseudo-horizon.

Note that in order to avoid a conical singularity at $r^2 = r_+^2$, the time coordinate t must be identified with the (real) period

$$\begin{aligned}\Delta t &= \frac{8\pi\sqrt{-B(r_+)}}{r_+\,V'(r_+)}\\ &= \frac{2\pi}{g(2 + g^2 r^2(H_1H_2 + H_1H_3 + H_2H_3))}\Big|_{r=r_+}\,.\end{aligned} \tag{3.42}$$

The metric is then geodesically complete, with r^2 ranging from $r^2 = r_+^2$ to ∞. The periodic coordinates t and ψ play the role of time in different regions. Outside the time machine, $r^2 > r_L^2$, t is the time coordinate, whilst inside the time machine, $r^2 < r_L^2$, it is ψ that is timelike.

The solution we have obtained has the topology $\mathbb{R}^2 \times S^3$, where the angular coordinate t and the radial variable $r2$ parameterise the $\mathbb{R}^2$ factor, while the Euler angles θ, ϕ and ψ parameterise the S^3. Strictly speaking, in order that the Euler coordinates not break down

near $r^2 = r_+^2$, one must introduce a shifted Euler coordinate $\psi' = \psi + f(r_+)\, t$. With this choice, the Euler angles θ, ϕ and ψ' have only the usual singularities on S^3, but they are globally defined for all r^2 and t. Because $r^2 = r_+^2$ is merely the centre of polar coordinates on the $\mathbb{R}^2$ factor, radial geodesics cannot reach values of r^2 less than r_+^2, and so the solution is of repulson type.

We now turn to an examination of the quantum mechanical consistency of these solutions. We begin by noting that the gauge potentials are not simultaneously well-defined at both the pseudo-horizon and at infinity. The gauge potentials given in (3.26) satisfy

$$A^i_\mu\, A^i_\nu\, g^{\mu\nu} = \frac{\alpha^2\, H_i(1+g^2 r^2 H_1 H_2 H_3 H_i^{-2}) - (H_i-1)^2 r^6\, H_1 H_2 H_3 H_i^{-1}}{r^6 H_i (H_1 H_2 H_3)^{2/3}\, V}\,, \tag{3.43}$$

which can be seen to diverge on the pseudo-horizon at $r^2 = r_+^2$, where V vanishes. On the other hand, at large r^2 we see from (3.43) that the A^i are non-singular, with the asymptotic behaviour

$$A^i_\mu\, A^i_\nu\, g^{\mu\nu} \sim \frac{\alpha^2\, g^2 - q_i^2}{g^2\, r^6}\,. \tag{3.44}$$

We can define gauge-transformed potentials

$$A^{i\prime} = A^i + c_i\, dt\,, \tag{3.45}$$

which are regular on the pseudo-horizon at $r^2 = r_+^2$, with the constants c_i given by

$$c_i \equiv \frac{2q_i + \alpha\, f(r_+)}{2 r_+^2\, H_i(r_+)}\,. \tag{3.46}$$

These potentials are however not pure gauge at infinity, *i.e.*

$$\oint A^{i\prime} \ne 0\,, \tag{3.47}$$

where the integral is taken over the closed timelike loop parameterised by t at infinity. This can also be seen from

$$A^{i\prime}{}_\mu\, A^{i\prime}{}_\nu\, g^{\mu\nu} \sim -\frac{c_i^2}{g^2\, r^2}\,, \tag{3.48}$$

as $r^2 \longrightarrow \infty$.

Requiring the quantum consistency of the fermion wave functions under the above gauge transformation implies that the phase factor

$$U = e^{\frac12\, {\rm i}\, g\, \sum_i c_i\, t} \tag{3.49}$$

should be single-valued. With the period of t given by (3.42), this implies the Josephson quantisation condition

$$\frac{g\,\Delta t}{8\pi\, r_+^2}\Big(\frac{2q_1+\alpha\, f}{H_1(r_+)} + \frac{2q_2+\alpha\, f}{H_2(r_+)} + \frac{2q_3+\alpha\, f}{H_3(r_+)}\Big) = \tilde n \in \mathbb{Z}\,, \tag{3.50}$$

and hence from (3.42)

$$\tilde{n} = \tfrac{1}{2} + \frac{1}{4 + 2g^2 r^2 (H_1 H_2 + H_1 H_3 + H_2 H_3)}\Big|_{r=r_+} . \tag{3.51}$$

This Josephson condition may be satisfied for appropriate choices of the parameters (bearing in mind that r_+^2 can be negative, provided that $r_+^2 + q_i > 0$ for all i in order to ensure that R^2 remains positive).

The Josephson quantisation condition can instead be derived by integrating the field strengths F^i over the $\mathbb{R}^2$ factor, as in (3.39), provided that one is careful in the definition of the $\mathbb{R}^2$ fibre. As we mentioned previously, in order to eliminate a coordinate singularity near the repulson, one must introduce the new Euler angle $\psi' = \psi + f(r_+)\,t$. The 1-forms A^i given in (3.26) now become

$$A^i = -\frac{1}{r^2 H_i}\Big(q_i dt - \tfrac{1}{2}\alpha(d\psi' + \cos\theta d\phi) - \tfrac{1}{2}\alpha\, f(r_+) dt\Big) . \tag{3.52}$$

We integrate $F^i = dA^i$ over the 2-surface defined by taking θ, ϕ and ψ' to be constant, yielding

$$\frac{g}{4\pi}\int_{\mathbb{R}^2}\sum_i F^i = \tfrac{1}{2} + \frac{1}{4 + 2g^2 r^2 (H_1 H_2 + H_1 H_3 + H_2 H_3)}\Big|_{r=r_+} . \tag{3.53}$$

We have seen that in general, a time machine implies the periodic identification of the time coordinate t, which implies an appropriate restriction of parameters to achieve quantum consistency. A special case arises, in which t does not need to be periodically identified, if V, considered as a function of r^2, has a double root. This can be achieved with the following choice of parameters:

$$\begin{aligned} &\alpha^2 = q_{(3)} + \tfrac{1}{2} q_{(2)} r_0^2 - \tfrac{1}{2} r_0^6 , \qquad g^2 = -\frac{2 r_0^2}{q_{(2)} + 2 q_{(1)} r_0^2 + 3 r_0^4} , \\ &q_{(1)} \equiv q_1 + q_2 + q_3 , \qquad q_{(2)} \equiv q_1 q_2 + q_1 q_3 + q_2 q_3 , \qquad q_{(3)} \equiv q_1 q_2 q_3 . \end{aligned} \tag{3.54}$$

Note that here $r_0^2 < 0$, with the constraint $r_0^2 + q_i > 0$ for all q_i. The function V is then given by

$$V = \frac{g^2 (r^2 - r_0^2)^2 (-q_{(2)} + 2 r_0^2 r^2 + r_0^4)}{2 r_0^2 r^4 (H_1 H_2 H_3)^{1/3}} . \tag{3.55}$$

Here, we present the full solution for the equal charge case:

$$\begin{aligned} ds^2 &= -\frac{R^2 V}{4B} dt^2 + \frac{dR^2}{V} + B(\sigma_3 + f\,dt)^2 + \tfrac{1}{4} R^2 (\sigma_1^2 + \sigma_2^2) , \\ V &= \frac{(R^2 - R_0^2)^2 (1 + 2g^2 R_0^2 + g^2 R^2)}{R^4} , \qquad B = \tfrac{1}{4} R^2 - \frac{R_0^6 (9 g^2 R_0^2 + 4)}{16 R^4} , \\ f &= \frac{2\alpha}{B R^2}\Big(1 - \frac{R_0^2 (3 g^2 R_0^2 + 2)}{2 R^2}\Big) , \qquad A^i_{(1)} = -\frac{1}{R^2}(q\, dt - \tfrac{1}{2}\alpha\sigma_3) , \\ \alpha^2 &= R_0^6 + \tfrac{9}{4} g^2 R_0^8 , \qquad q = R_0^2 + \tfrac{3}{2} g^2 R_0^4 . \end{aligned} \tag{3.56}$$

The solution describes a BMPV type of repulson, for which the t coordinate is not periodically identified. One again needs to make gauge transformations to define new potentials that are regular on the pseudo-horizon; these are the same as in the case with general rotation that we discussed previously, with the specialisation of parameters given in (3.54). Since there is no periodic identification of t in this case, no quantum-consistency condition arises. In fact, this case corresponds to the situation where the denominator in (3.51) vanishes, and so it is associated with the $\tilde{n} \to \infty$ limit of the solutions with periodically-identified time.

Finally, for values of the parameters for which $r_+^2 + q_i < 0$, there will be naked singularities.

3.4 $E - 2gJ - \sum Q_i = 0$: Case B

The cases $E - 2gJ - \sum Q_i = 0$ and $E + 2gJ - \sum Q_i = 0$ are equivalent, modulo a reversal of the sign of a, and the discussion for the two is equivalent. We shall consider the former. As can be seen from (3.16), the Bogomolny matrix will then have one zero eigenvalue, and so the solution preserves $\frac{1}{4}$ of the $\mathcal{N} = 2$ supersymmetry. To satisfy the supersymmetric condition

$$e^{\delta_1+\delta_2+\delta_3} = \frac{1}{ag}\,, \tag{3.57}$$

it is most convenient to use it to express the parameter a in terms of δ_i. The general solution is somewhat cumbersome to present, and we shall first discuss the situation where the three charges are set equal.

Making the variable changes

$$m = \frac{2\sqrt{g\,\alpha}q^{3/2}}{g\,\alpha - q}\,,\qquad e^{2\delta} = \sqrt{q/(g\,\alpha)}\,,\qquad r^2 = R^2 - \frac{(\sqrt{g\,\alpha} - \sqrt{q})^2}{g\,\alpha - q}\,, \tag{3.58}$$

the solution becomes

$$\begin{aligned}
ds_5^2 &= -\frac{R^2 V}{4B}\,dt^2 + B(\sigma_3 + f\,dt)^2 + \frac{dR^2}{V} + \tfrac{1}{4}R^2(\sigma_1^2 + \sigma_2^2)\,,\\
A^i &= -\frac{1}{R^2}\,(q dt - \tfrac{1}{2}\alpha\,\sigma_3)\,,\\
V &= 1 + g^2R^2 + \frac{2(q + 2\alpha g)}{R^2} - \frac{(g\alpha + q)(g\alpha - q)^2 - 4\alpha^2}{(g\alpha - q)R^4}\,,\\
B &= \tfrac{1}{4}R^2 + \frac{\alpha^2}{(g\alpha - q)R^2} - \frac{\alpha^2}{4R^4}\,,\\
f &= \frac{\alpha(g\alpha\,q - q^2 - g\alpha R^2 - 3qR^2)}{2(g\alpha - q)R^4 B}\,.
\end{aligned} \tag{3.59}$$

The thermodynamic quantities for this solution are given by

$$E = \frac{\pi(3q^2 + 3g\alpha\, q + 2g^2\alpha^2)}{4(g\alpha - q)}\,, \qquad J = \frac{\pi\alpha(g\alpha + 3q)}{4(g\alpha - q)}\,, \qquad Q = -\tfrac{1}{4}\pi q\,. \tag{3.60}$$

These local solutions were obtained in [16, 17] using different variables. Here, we shall discuss in some detail their global structure.

First, it is easy to see that ∂_t cannot have a spinorial square root, since $g_{tt} = g(K, K)$ can become positive. As discussed previously, the coordinate transformation $\psi \to \psi + 2g\, t$, $t \to t$, yields $K_+ = \partial_t + 2g\, \partial_\psi$ as the Killing vector with a spinorial square root. In fact, we have

$$g(K_+, K_+) = -\frac{VR^2}{4B} + B(f + 2g)^2 = -\frac{(R^2 + q - g\alpha)^2}{R^4}\,. \tag{3.61}$$

This equation shows that if there is a Killing horizon at $r = r_+ > 0$ where $V(r_+) = 0$, then $B(r_+)$ must be negative unless either

(i) The right-hand side of the above equation goes to zero at the horizon, or

(ii) $B(r_+) = 0$, *i.e.* the VLS coincides with the horizon.

Note that the first case is never possible for the $\frac{1}{2}$-supersymmetric solution described in section 3.3. Here, however, it can be achieved, by choosing the parameters q and α to be

$$q = -R_0^2 - \tfrac{1}{2}g^2 R_0^4\,, \qquad \alpha = -\tfrac{1}{2}gR_0^4\,, \tag{3.62}$$

whereupon the solution becomes

$$\begin{aligned} V &= \frac{(R^2 - R_0^2)^2(g^2R^2 + 2g^2R_0^2 + 1)}{R^4}\,, \qquad B = \tfrac{1}{4}R^2 + \frac{g^2R_0^6}{4R^2} - \frac{g^2R_0^8}{16R^4}\,, \\ f &= \frac{1}{4B}\Big(\frac{gR_0^6(g^2R_0^2 + 2)}{2R^4} - \frac{gR_0^4(2g^2R_0^2 + 3)}{R^2}\Big)\,. \end{aligned} \tag{3.63}$$

In this case the Killing horizon coincides with an event horizon, and the VLS occurs inside this horizon. The solution, which was obtained in [23], describes a supersymmetric black hole that is regular outside and on the event horizon.

Alternatively, we can also avoid naked CTC's by considering the possibility (ii) listed above, where the VLS occurs on the horizon. This can be achieved by choosing the parameters so that

$$q = \frac{\alpha^2}{R_0^4}\,, \qquad g = \frac{\alpha(\alpha^2 + 3R_0^6)}{R_0^4(\alpha^2 - R_0^6)}\,. \tag{3.64}$$

Note that in this case, it is more convenient to express g in terms of α. The metric functions are given by

$$\begin{aligned}
V &= g^2(R^2-R_0^2)\Big(1+\frac{R_0^2(R_0^6+\alpha^2)(R_0^{12}+6\alpha^2R_0^6+\alpha^4)}{\alpha^2R^2(3R_0^6+\alpha^2)^2} \\
&\qquad +\frac{R_0^4(R_0^{18}-3\alpha^2R_0^{12}+11\alpha^4R_0^6+7\alpha^6)}{\alpha^2R^4(3R_0^6+\alpha^2)^2}\Big)\,, \\
B &= \tfrac14(R^2-R_0^2)(1+\frac{R_0^2}{R^2}+\frac{\alpha^2}{R_0^2R^4})\,, \\
f &= -\frac{2\alpha^3}{R_0^4(R_0^2R^4+R_0^4R^2+\alpha^2)}\,.
\end{aligned} \tag{3.65}$$

This solution describes a regular soliton, *i.e.* with no horizon and no CTC's. To avoid a conical singularity at $R=R_0$, the quantisation condition

$$\frac{(2R_0^6+\alpha^2)(R_0^6+3\alpha^2)}{2R_0^6(R_0^6-\alpha^2)}=k \tag{3.66}$$

must be satisfied, where k is the integer characterising the topology of the $S^3/\mathbb{Z}_k$ spatial sections.

Aside from the two special cases enumerated above, all the remaining solutions will have naked CTC's. Among these, there are two possibilities. One is that the metric has no Killing horizon, and hence we have a naked singularity cloaked by a VLS. The other possibility is that the metric has a pseudo-horizon inside the VLS. This situation is very much like the $\frac12$-supersymmetric solutions, which we obtained in section 3.3. In general, it is necessary to identify the t coordinate periodically, to avoid a conical singularity at the horizon. The geodesics are complete from the horizon to infinity. Since the pseudo-horizon lies inside the VLS, it is a repulson.

A special case of the latter category is where there is a pseudo-horizon with a double root that lies inside the VLS. In this case, regularity on the pseudo-horizon does not require making a periodic identification of the t coordinate. The solution is achieved by choosing the parameters so that

$$q=\frac{(9g^2R_0^2+8)(3g^2R_0^2+2)}{18g^2}\,,\qquad \alpha=-\frac{(9g^2R_0^2+4)(3g^2R_0^2+2)}{18g^3}\,. \tag{3.67}$$

We find that the metric functions become

$$\begin{aligned}
V &= \frac{(R^2-R_0^2)^2(g^2R^2+2g^2R_0^2+1)}{R^4}\,, \\
B &= \tfrac14R^2-\frac{(9g^2R_0^2+4)^2}{108g^4R^2}\Big(1+\frac{(3g^2R_0^2+2)^2}{12g^2R^2}\Big)\,, \\
f &= -\frac{(9g^2R_0+4)}{648g^5B}\Big(\frac{6g^2(9g^2R_0^2+10)}{R^2}+\frac{(9g^2R_0^2+8)(3g^2R_0^2+2)^2}{R^4}\Big)\,,
\end{aligned} \tag{3.68}$$

from which we see that V has a double root at $R = R_0$. However, the function B at $R = R_0$ becomes

$$B = -\frac{(9g^2 R_0^2 + 2)^2 (3g^2 R_0^2 + 4)^2}{1296 g^6 R_0^4} \,. \tag{3.69}$$

Since this is negative, it implies the occurrence of CTC's outside the pseudo-horizon. The structure of this solution is analogous to the BMPV repulson.

We now turn to the general situation (3.3) where the three charges are unequal, but satisfy (3.57), Using a similar analysis to that for the three equal charges, we first examine $g(K_+, K_+)$, given by

$$\begin{aligned} g(K_+, K_+) &= -\frac{(H_1 H_2 H_3)^{1/3}}{f_1} \Big(r^2 Y - \frac{(f_2 - g f_1)^2}{r^4 H_1 H_2 H_3} \Big) \\ &= -(H_1 H_2 H_3)^{-2/3} \Big(1 - \frac{m(e^{-2\delta_1} + e^{-2\delta_2} + e^{-2\delta_3} - e^{-2\delta_1 - 2\delta_2 - 2\delta_3} - 2)}{2r^2} \Big)^2 \,. \end{aligned} \tag{3.70}$$

The non-positivity of this quantity implies that in general there is a pseudo-horizon inside the VLS. There are two cases where naked CTC's can be avoided. One case arises if the horizon coincides with the the VLS. This can be achieved by taking

$$m\, g^2 = -\frac{2(1 + e^{2\delta_1 + 2\delta_2} + e^{2\delta_1 + 2\delta_3} + e^{2\delta_2 + 2\delta_3})^2}{(e^{4\delta_1} - 1)(e^{4\delta_2} - 1)(e^{4\delta_3} - 1) e^{2(\delta_1 + \delta_2 + \delta_3)}} \,. \tag{3.71}$$

Then, the functions Y and f_1 have the same root $r = r_0$, given by

$$g^2\, r_0^2 = \frac{2(1 + e^{2\delta_1 + 2\delta_2} + e^{2\delta_1 + 2\delta_3} + e^{2\delta_2 + 2\delta_3})}{(e^{2\delta_1} + 1)(e^{2\delta_2} + 1)(e^{2\delta_3} + 1) e^{2(\delta_1 + \delta_2 + \delta_3)}} \,. \tag{3.72}$$

The solution describes a topological soliton, running from $r^2 = r_0^2$, which has spatial sections that are topologically an R^2 bundle over S^2, to AdS_5 at infinity. The charge parameters must satisfy a quantisation condition in order to avoid a conical singularity associated with the collapsing of σ_3; this condition is given by

$$\begin{aligned} &\frac{(e^{2\delta_1} + e^{2\delta_2} + e^{2\delta_3} + e^{2(\delta_1 + \delta_2 + \delta_3)})(1 - e^{4\delta_1 + 4\delta_2} - e^{4\delta_1 + 4\delta_3} - e^{4\delta_2 + 4\delta_3} + 2e^{4(\delta_1 + \delta_2 + \delta_3)}) r_0}{(e^{4\delta_1} - 1)(e^{4\delta_2} - 1)(e^{4\delta_3} - 1) e^{2(\delta_1 + \delta_2 + \delta_3)}} \\ &= k \,, \end{aligned} \tag{3.73}$$

for $S^3/\mathbb{Z}_k$ spatial sections.

The alternative way to avoid naked CTC's is by ensuring that the right-hand side of (3.70) is zero at the Killing horizon. This requires that the function Y have a root $r^2 = r_0^2$ given by

$$r_0^2 = \tfrac{1}{2} m (e^{-2\delta_1} + e^{-2\delta_2} + e^{-2\delta_3} - e^{-2\delta_1 - 2\delta_2 - 2\delta_3} - 2) \,, \tag{3.74}$$

which implies that

$$m = \frac{1}{2g^2 \sinh(\delta_1 + \delta_2) \sinh(\delta_1 + \delta_3) \sinh(\delta_2 + \delta_3)} \,. \tag{3.75}$$

Remarkably, $r^2 = r_0^2$ is a double root for Y. For the solution to be free of naked singularities, it is necessary that $r_+^2 H_i > 0$ for all $i = 1, 2, 3$, which places restrictions on the domain of allowed charge parameters δ_i. The resulting regular black hole solutions were previously obtained in [24]. The near-horizon geometry is a direct product of AdS_2 and a squashed 3-sphere. The timelike Killing field coincides with an everywhere-causal Killing field on AdS_2, and the system has zero temperature. The time machine is located strictly inside the horizon.

For solutions other than the two examples discussed above, naked CTC's are inevitable, implying the existence of time machines. Depending on the parameters, there can be naked singularities, or regular solutions with t being periodically identified, or else BMPV type repulsons, where t is not periodic and Y has a double root.

3.5 Lifting to type IIB supergravity

All the solutions we considered above can be lifted back to become solutions of $D = 10$ type IIB supergravity. The reduction ansatz for the five-dimensional $U(1)^3$ gauged supergravity theory can be found in [40]. The metric ansatz is given by [40]

$$ds_{10}^2 = \sqrt{\Delta} ds_5^2 + \frac{1}{g^2 \sqrt{\Delta}} \sum_{i=1}^{3} X_i^{-1} \Big(d\mu_i^2 + \mu_i^2 (d\phi_i + g A_{(1)}^i)^2 \Big) . \tag{3.76}$$

where $\Delta = \sum X_i \, \mu_i^2$. With this explicit reduction ansatz, global properties such as the periods, and CTC's, can be now addressed from the ten-dimensional point of view.

In general, the ten-dimensional spacetime is an S^5 bundle over the five-dimensional spacetime, with structural group $U(1) \times U(1) \times U(1)$, where the i'th $U(1)$ acts on S^5 by advancing the angle ϕ_i. The requirement that the ten-dimensional metric extend smoothly onto a non-singular ten-dimensional manifold leads to the conditions

$$\frac{g}{2\pi} \int_C F^i \in \mathbb{Z} \,, \tag{3.77}$$

where C is any non-trivial 2-cycle in the five-dimensional spacetime, since the azimuthal coordinates ϕ_i are constrained by the regularity of the S^5 to have periods 2π.

For the topological soliton solutions in section 3.3.1, we find

$$\frac{g}{2\pi} \in\in^{S^2} F^i = -\frac{g\,\alpha}{q_i} \,, \qquad \frac{g}{2\pi} \in\in^{\mathbb{R}^2} F^i = -\frac{g\,\alpha}{k\, q_i} \,. \tag{3.78}$$

In view of the fact that all the q_i are necessarily strictly positive, it follows from (3.32) that the integrals over $\mathbb{R}^2$ can never satisfy the conditions (3.77).

For the time machines discussed in section 3.3.2, we find that for appropriate choices of the parameters we can satisfy the consistency conditions (3.77). For example, if we set the charges q_i equal for simplicity, then (3.77) will be satisfied if the integer $\tilde{n}$ in (3.51) is a multiple of 3.

It is straightforward to verify that for the $\frac{1}{2}$ supersymmetric time machine described in section 3.3, the five-dimensional CTC's associated with ψ are no longer CTC's in ten dimensions, since

$$g^{(10)}_{\psi\psi} = \tfrac{1}{2} R^2 \sqrt{\Delta}\,, \tag{3.79}$$

which is positive definite. However, CTC's do still exist in ten dimensions. This can be seen by examining the determinant of the sub-metric involving the angular coordinates $(\phi_1, \phi_2, \phi_3, \psi \equiv \phi_4)$, which is given by

$$\det(g_{ij}) = \frac{B\mu_1^2\mu_2^2\mu_3^2}{g^6\Delta}\,. \tag{3.80}$$

This is negative in the region where B is negative, showing that CTC's are inevitable in ten dimensions, if they exist in five dimensions (this was seen in the case of the solutions with CTC's discussed in section 3.3 in [41]).

4 Rotating Black Holes and Supersymmetric Limits in Seven-Dimensional Gauged Supergravity

4.1 Black-Hole Thermodynamics in Seven Dimensions

Rotating black holes in seven-dimensional gauged supergravity are somewhat more complicated than those in five dimensions. One reason for this is that there is a "first-order self-duality" equation for the 4-form field in the seven-dimensional theory, and this plays a non-trivial role in the solutions that were obtained in [19]. The relevant part of the Lagrangian for $SO(5)$ gauged supergravity in seven dimensions, in which only the fields that are non-zero in the solutions are retained, is given by

$$\begin{aligned}
\mathcal{L}_7 \;=\;& R\,{*\mathbb{1}} - \tfrac{1}{2}{*d\varphi_i} \wedge d\varphi_i - \tfrac{1}{2}\sum_{i=1}^{2} X_i^{-2}\, {*F^i_{(2)}} \wedge F^i_{(2)} - \tfrac{1}{2}(X_1\, X_2)^2\, {*F_{(4)}} \wedge F_{(4)} \\
& -2g^2\left[(X_1\, X_2)^{-4} - 8X_1\, X_2 - 4X_1^{-1} X_2^{-2} - 4X_1^{-2}\, X_2^{-1}\right] {*\mathbb{1}} \\
& -g\, F_{(4)} \wedge A_{(3)} + F^1_{(2)} \wedge F^2_{(2)} \wedge A_{(3)}\,,
\end{aligned} \tag{4.1}$$

where

$$F^i_{(2)} = dA^i_{(1)}\,, \qquad F_{(4)} = dA_{(3)}\,,$$

$$X_1 = e^{-\frac{1}{\sqrt{2}}\varphi_1 - \frac{1}{\sqrt{10}}\varphi_2}\,, \qquad X_2 = e^{\frac{1}{\sqrt{2}}\varphi_1 - \frac{1}{\sqrt{10}}\varphi_2}\,, \tag{4.2}$$

together with the first-order odd-dimensional self-duality equation to be imposed after the variation of the Lagrangian. This condition is conveniently stated by introducing an additional 2-form potential $A_{(2)}$, which can be gauged away in the gauged theory, and defining

$$F_{(3)} = dA_{(2)} - \tfrac{1}{2} A^1_{(1)} \wedge dA^2_{(1)} - \tfrac{1}{2} A^2_{(1)} \wedge dA^1_{(1)}\,. \tag{4.3}$$

The odd-dimensional self-duality equation then reads

$$(X_1\, X_2)^2 *F_{(4)} = -2g\, A_{(3)} - F_{(3)}\,. \tag{4.4}$$

The non-extremal rotating black hole solutions found in [19] are given by

$$\begin{aligned}
ds_7^2 &= (H_1 H_2)^{1/5}\Big[-\frac{Y\,dt^2}{f_1\,\Xi_-^2} + \frac{r^2\,\rho^4\,dr^2}{Y} + \frac{f_1}{\rho^4\,H_1 H_2\,\Xi^2}\Big(\sigma - \frac{2 f_2}{f_1}\,dt\Big)^2 + \frac{r^2+a^2}{\Xi}\,d\Sigma_2^2\Big]\,,\\
A^i_{(1)} &= \frac{2m\,s_i}{\rho^4\,\Xi\,H_i}\,(\alpha_i\,dt + \beta_i\,\sigma)\,,\\
A_{(2)} &= \frac{m\,a\,s_1\,s_2}{\rho^4\,\Xi_-^2}\Big(\frac{1}{H_1} + \frac{1}{H_2}\Big)\,dt\wedge\sigma\,, \qquad A_{(3)} = \frac{2m\,a\,s_1\,s_2}{\rho^2\,\Xi\,\Xi_-}\,\sigma\wedge J\,,\\
X_i &= (H_1 H_2)^{2/5}\,H_i^{-1}\,, \qquad H_i = 1 + \frac{2m\,s_i^2}{\rho^4}\,, \qquad \rho^2 = (r^2+a^2)\,,\\
\alpha_1 &\equiv c_1 - \tfrac{1}{2}(1-\Xi_+^2)(c_1-c_2)\,, \qquad \alpha_2 \equiv c_2 + \tfrac{1}{2}(1-\Xi_+^2)(c_1-c_2)\,,\\
\beta_1 &= -a\,\alpha_2\,, \qquad \beta_2 = -a\,\alpha_1\,,\\
\Xi_\pm &\equiv 1 \pm a\,g\,, \qquad \Xi \equiv 1 - a^2\,g^2 = \Xi_-\,\Xi_+\,,
\end{aligned} \tag{4.5}$$

where the functions f_1, f_2 and Y are given by

$$\begin{aligned}
f_1 &= \Xi\,\rho^6\,H_1 H_2 - \frac{4\Xi_+^2\,m^2\,a^2\,s_1^2\,s_2^2}{\rho^4} + \tfrac{1}{2}m\,a^2\Big[4\Xi_+^2 + 2c_1\,c_2\,(1-\Xi_+^4) + (1-\Xi_+^2)^2\,(c_1^2+c_2^2)\Big]\,,\\
f_2 &= -\tfrac{1}{2}g\,\Xi_+\,\rho^6\,H_1 H_2 + \tfrac{1}{4}m\,a\Big[2(1+\Xi_+^4)\,c_1\,c_2 + (1-\Xi_+^4)\,(c_1^2+c_2^2)\Big]\,,\\
Y &= g^2\,\rho^8\,H_1 H_2 + \Xi\,\rho^6 + \tfrac{1}{2}m\,a^2\Big[4\Xi_+^2 + 2(1-\Xi_+^4)\,c_1\,c_2 + (1-\Xi_+^2)^2(c_1^2+c_2^2)\Big]\\
&\quad -\tfrac{1}{2}m\,\rho^2\Big[4\Xi + 2a^2g^2(6+8ag+3a^2g^2)\,c_1\,c_2 - a^2g^2(2+ag)(2+3ag)(c_1^2+c_2^2)\Big]\,.
\end{aligned} \tag{4.6}$$

The metric $d\Sigma_2^2$ is the standard Fubini-Study metric on $\mathbb{CP}^2$, and $\sigma = d\psi + \mathcal{B}$, where $\frac{1}{2}d\mathcal{B}$ is the Kähler form on $\mathbb{CP}^2$. The coordinate ψ, which has period 2π, lives on the $U(1)$ fibre of S^5 viewed as a $U(1)$ bundle over $\mathbb{CP}^2$.

In these solutions the three *a priori* independent angular momenta have been set equal. There are two independent electric charges, characterised by $s_i = \sinh\delta_i$ (as usual we are also using the notation $c_i = \cosh\delta_i$). These charges are carried by the two 2-form fields of the $U(1)\times U(1)$ subgroup of $SO(5)$.

We again follow the strategy of [29] in order to evaluate the energy E of the black hole solution, by integrating up the first law of thermodynamics. First, we note that the time coordinate t in (4.5) has a non-canonical normalisation, as measured at infinity, and so we introduce the canonically-normalised $\tilde{t}$, defined by $t = \tilde{t}\,\Xi_-$. The Killing vector $\partial/\partial\tilde{t}$ is rotating at infinity. In fact we have precisely $K_+ = \partial/\partial\tilde{t}$, where K_+ has a spinorial square root near infinity, as discussed in section 2.2. We can pass to non-rotating coordinates by defining a new $U(1)$ fibre coordinate $\psi' = \psi + g\,\tilde{t}$. After performing these transformations, we find that the entropy, temperature, angular velocity and electrostatic potentials on the horizon are given by

$$\begin{aligned} S &= \frac{\pi^3\,(r^2+a^2)\,\sqrt{f_1}}{\Xi^3}\,, \qquad T = \frac{Y'}{8\pi\,r\,(r^2+a^2)^{3/2}\,\sqrt{f_1}}\,, \qquad \Omega = g + \frac{2f_2\,\Xi_-}{f_1}\,, \\ \Phi_i &= \frac{2ms_i}{\rho^4\,\Xi\,H_i}\,[\alpha_i + \beta_i(\Omega - g)]\,, \end{aligned} \tag{4.7}$$

where all functions are evaluated at the outer horizon $r = r_+$ where $Y(r)$ has its largest positive root.

Again the evaluation of the electric charges and the angular momentum is straightforward using integrals over the S^5 at infinity. After calculations of some complexity, we are then able to obtain the energy by integration of the first law, which for this case reads

$$dE = T\,dS + 3\Omega\,dJ + \sum_i \Phi_i\,dQ_i\,. \tag{4.8}$$

Our results are

$$\begin{aligned} E &= \frac{m\,\pi^2}{32\Xi^4}\Big[12\Xi_+^2\,(\Xi_+^2-2) - 2c_1c_2\,a^2g^2\,(21\Xi_+^4 - 20\Xi_+^3 - 15\Xi_+^2 - 10\Xi_+ - 6) \\ &\qquad\qquad + (c_1^2+c_2^2)(21\Xi_+^6 - 62\Xi_+^5 + 40\Xi_+^4 + 13\Xi_+^2 - 2\Xi_+ + 6)\Big]\,, \\ J &= \frac{m\,a\,\pi^2}{16\Xi^4}\Big[4ag\Xi_+^2 - 2c_1c_2(2\Xi_+^5 - 3\Xi_+^4 - 1) + ag(c_1^2+c_2^2)(\Xi_+ + 1)(2\Xi_+^3 - 3\Xi_+^2 - 1)\Big]\,, \\ Q_1 &= \frac{m\,\pi^2\,s_1}{4\Xi^3}\Big[a^2g^2c_2(2\Xi_+ + 1) - c_1(2\Xi_+^3 - 3\Xi_+^2 - 1)\Big]\,, \\ Q_2 &= \frac{m\,\pi^2\,s_2}{4\Xi^3}\Big[a^2g^2c_1(2\Xi_+ + 1) - c_2(2\Xi_+^3 - 3\Xi_+^2 - 1)\Big]\,. \end{aligned} \tag{4.9}$$

In this case, the fact that the right-hand side of (4.8) turns out to be an exact form, allowing integration to give the energy function E, is highly non-trivial, and it provides a striking demonstration of the validity of the first law of thermodynamics for these seven-dimensional rotating black hole solutions.

4.2 Supersymmetric Limits and the Supersymmetric Bound

The algebra of the supercharges $\mathcal{Q}$ in seven-dimensional $\mathcal{N}=2$ gauged AdS supergravity is given by

$$M \equiv \{\mathcal{Q}, \overline{\mathcal{Q}}\} = \tfrac{1}{2} J_{AB}\, \gamma^{AB} + Z\,, \tag{4.10}$$

where J_{07} is the energy, and J_{ij} for $1 \le i \le 6$ correspond to angular momenta. If there are three parameters J_1, J_2 and J_3 describing rotations in the J_{12}, J_{34} and J_{56} planes, we may write

$$J_{07} = g^4\, E\,, \qquad J_{12} = g^5\, J_1\,, \qquad J_{34} = g^5\, J_2\,, \qquad J_{56} = g^5\, J_3\,, \qquad Z = g^4 \sum_i Q_i\,. \tag{4.11}$$

(See [34] for a discussion of the supersymmetry algebra, and [42] for a discussion of supersymmetric non-rotating AdS black holes in seven dimensions.) The eigenvalues of the Bogomolny matrix $g^{-4}\, M$ acting on chiral eigenspinors are given by

$$\begin{aligned}
\lambda &= E + gJ_1 - gJ_2 - gJ_3 - \sum_i Q_i\,, \qquad &&\text{and 2 cyclic}\,,\\
\lambda &= E + gJ_1 - gJ_2 - gJ_3 + \sum_i Q_i\,, \qquad &&\text{and 2 cyclic}\,,\\
\lambda &= E + gJ_1 + gJ_2 + gJ_3 - \sum_i Q_i\,, &&\\
\lambda &= E + gJ_1 + gJ_2 + gJ_3 + \sum_i Q_i\,, &&
\end{aligned} \tag{4.12}$$

where on the first two lines there are two further eigenvalues corresponding to cycling the $+$ sign onto J_2 or J_3 instead of J_1.

If we set the three angular momenta equal, $J_1 = J_2 = J_3 = J$, we get

$$\begin{aligned}
\lambda &= E - gJ - \sum_i Q_i \qquad &&\text{thrice}\,,\\
\lambda &= E - gJ + \sum_i Q_i \qquad &&\text{thrice}\,,\\
\lambda &= E + 3gJ - \sum_i Q_i \qquad &&\text{once}\,,\\
\lambda &= E + 3gJ + \sum_i Q_i \qquad &&\text{once}\,.
\end{aligned} \tag{4.13}$$

Substituting our expressions (4.9) for the energy, angular momentum and charges of the rotating black holes (4.5), we find that the four cases in (4.13), we find that the vanishing

of λ is achieved, respectively, if[8]

$$e^{\delta_1+\delta_2} = \quad 1+\frac{2}{ag}\,, \quad 1-\frac{2}{ag}\,, \quad 1-\frac{2}{3ag}\,, \quad 1+\frac{2}{3ag}\,. \tag{4.14}$$

Thus we are led to supersymmetric limits of the non-extremal rotating black holes (4.5) when

$$e^{\delta_1+\delta_2} = 1\pm\frac{2}{ag}\,, \tag{4.15}$$

which preserve $\frac{3}{8}$ of the supersymmetry, and which we designate as type A. Also, we obtain supersymmetric limits when

$$e^{\delta_1+\delta_2} = 1\pm\frac{2}{3ag}\,, \tag{4.16}$$

which preserve $\frac{1}{8}$ of the supersymmetry, and which we designate as type B. As we shall show below, the former in general all have closed timelike curves, while the latter include a particular case, for a special choice of the parameters, which gives a perfectly regular supersymmetric black hole with an horizon.

The general analysis of the convex cone implied by the positivity of the Bogomolny matrix eigenvalues described in section 3.2 applies to this $D = 7$ case as well. There are three angular momenta and so the cone lies in $\mathbb{R}^5$ and is bounded by eight hyperplanes. If $J_1 = J_2 = J_3$, then two sets of three of these four-dimensional hyperplanes or faces intersect on two-dimensional faces of enhanced supersymmetry. Again, states on the faces are what we call type B, while states on the edges are what we call type type A.

4.3 $E - g\,J - \sum_i Q_i = 0$: Case A

As we saw previously, this supersymmetry condition is satisfied provided that

$$e^{\delta_1+\delta_2} = 1+\frac{2}{ag}\,. \tag{4.17}$$

In general, the solution describes a naked time machine. This can be seen by examining the component of the metric

$$\begin{aligned} g(K_+, K_+) &= \frac{1}{f_1}\Big(\frac{4f_2^2}{R^4 H_1 H_2 \Xi^2} - \frac{Y}{\Xi_-^2}\Big) \\ &= -(1+ag)^2 R^4 - \tfrac{1}{8} m e^{-2\delta_1-2\delta_2}(ag\,e^{\delta_1+\delta_2} - 2 - ag)((2+ag)e^{\delta_1+\delta_2} - ag) \\ &\qquad \times((e^{\delta_1} - e^{\delta_2})ag + 2e^{\delta_1})((e^{\delta_1} - e^{\delta_2})ag + 2e^{\delta_2})\,, \end{aligned} \tag{4.18}$$

[8] Again, as in the five-dimensional case, there are spurious roots (which in this case correspond to quite complicated relations between a, g and the δ_i), which do not correspond to supersymmetric solutions within the class of metrics we are considering.

When the supersymmetry condition is satisfied, the second term vanishes, and hence

$$g(K_+,K_+) = -(1+ag)^2 R^4\,, \tag{4.19}$$

which is negative definite for all R outside the singularity at $R=0$. Near the horizon, where Y approaches zero, it follows that f_1 must be negative, which implies the occurrence of naked CTC's.

To discuss the global structure in detail, we first consider the solution with two equal charges, corresponding to $\delta_1=\delta_2=\delta$. Making the changes of the variables and coordinates

$$R=\sqrt{\Xi}r\,,\qquad t=\Xi_-\,\tilde t\,,\qquad q=\frac{2m\sinh^2\delta}{\Xi^2}\,,\qquad \alpha=-\frac{aq}{1-ag}\,, \tag{4.20}$$

and imposing the supersymmetry condition $e^{2\delta}=1+2/(ag)$, we find that the solution (4.5) becomes

$$\begin{aligned}
ds^2 &= H^{2/5}\Big(-\frac{V}{H^2B}r^2d\tilde t^2+B(\sigma+f\,d\tilde t)^2+\frac{dr^2}{V}+r^2d\Sigma_2^2\Big)\\
A^1_{(1)} &= A^2_{(1)}=\frac{1}{r^4H}(q\,d\tilde t+\alpha\sigma)\,,\quad A_{(2)}=-\frac{\alpha}{r^4H}\,d\tilde t\wedge\sigma\,,\quad A_{(3)}=-\frac{\alpha}{r^2}\sigma\wedge J\,,\\
X_1 &= X_2=H^{-1/5}\,,\qquad H=1+\frac{q}{r^4}\,,\\
V &= 1+g^2r^2H^2+\frac{2g\,\alpha}{r^4}\,,\quad B=r^2-\frac{\alpha^2}{r^8H^2}\,,\quad f=\frac{1}{B}\Big(gr^2+\frac{\alpha}{r^4H^2}\Big)\,.
\end{aligned} \tag{4.21}$$

Note that

$$g(K_+,K_+)=H^{2/5}\Big(-\frac{Vr^2}{H^2B}+Bf^2\Big)=-H^{-8/5} \tag{4.22}$$

is negative definite. This rules out the possibility of a black hole without naked CTC's. From (4.9), the thermodynamic quantities are now given by

$$E=\tfrac18\pi^2(4q-5g\alpha)\,,\qquad J=-\tfrac18\pi^2\alpha\,,\qquad Q_1=Q_2=\tfrac14\pi^2(q-g\alpha)\,. \tag{4.23}$$

The metric behaviour depends on the sign of the rotation parameter α. If α is positive, it is clear that the solution has a naked singularity at $r=0$, cloaked by the VLS at $r=r_L$ where $B(r_L)$ vanishes. If instead α is sufficiently negative, the solution will develop a Killing horizon at $r=r_+<r_L$, inside the VLS. To avoid a conical singularity at this pseudo-horizon, it is necessary that the $\tilde t$ coordinate be periodic, with period given by

$$\Delta\tilde t=\frac{\pi(g^2(r_+^4+q)^2-r_+^6)}{g^3(3r_+^8+2qr_+^4-q^2)+2gr_+^6}\,. \tag{4.24}$$

It is also possible for V to have a double root, in which case the $\tilde t$ coordinate does not require a periodic identification. This occurs if the parameters satisfy the conditions

$$g^2=\frac{2r_0^6}{(q+r_0^4)(q-3r_0^4)}\,,\qquad g\alpha=-\frac{r_0^4(3q-r_0^4)}{2(q-3r_0^4)}\,. \tag{4.25}$$

The function V in the metric (4.21) is then given by

$$V = \frac{(r^2-r_0^2)^2}{g^2 r^2}\Big(1+\frac{(q-r_0^4)^2}{2r_0^6 r^2}+\frac{q^2}{r_0^4 r^4}\Big)\,. \tag{4.26}$$

The remaining details of the solution can be obtained by substituting the α and g parameters given above. At $r = r_0$, we have

$$B = -\frac{g^2(q+5r_0^4)^2}{16 r_0^4}\,, \tag{4.27}$$

which is negative, implying the occurrence of naked CTC's.

Let us now consider the case of a critical rotation, such that the solution contains no naked CTC's. This can be achieved if V and B approach zero simultaneously, which occurs if

$$\alpha = -\frac{r_0^4}{g}\,, \qquad q = g^{-1} r_0^3 - r_0^4\,. \tag{4.28}$$

The metric functions are then given by

$$\begin{aligned}
V &= (r^2-r_0^2)\Big[g^2+\frac{1+g^2 r_0^2}{r^2}-\frac{r_0^2(g^2r_0^2-2gr_0-1)}{r^4}-\frac{r_0^4(gr_0-1)^2}{r^6}\Big]\,,\\
B &= \frac{r^2-r_0^2}{g^2 r^8 H^2}\Big[(r_0^3+gr_0^2r^2+gr^4)^2-g^2r_0^2r^2(r^2+r_0^2)^2\Big]\,,\\
f &= \frac{(r^2-r_0^2)}{gr^6\, BH^2}\Big[g^2r^2(r^2+r_0^2)^2-r_0^2(gr^2+gr_0^2-r_0)^2\Big]\,.
\end{aligned} \tag{4.29}$$

We must then examine the metric in the neighbourhood of $r = r_0$, where $g_{\psi\psi} \to 0$, in order to determine the conditions for regularity. Defining $r - r_0 = \rho^2$, we find that the metric near $\rho = 0$ becomes

$$ds^2 \sim H(r_0)^{2/5}\Big[\frac{2r_0}{1+4gr_0}\Big(d\rho^2+(1+4gr_0)^2\rho^2(\sigma+f\,dt)^2\Big)+\cdots\Big]\,. \tag{4.30}$$

Since ψ has period $2\pi/k$ for $S^5/\mathbb{Z}_k$, it follows that the quantisation condition

$$1+4gr_0 = k \tag{4.31}$$

must hold. With this condition, we obtain a completely regular topological soliton, analogous to the five-dimensional example that we found section 3.3.

It is worth remarking that in this case, we have

$$H = 1+\frac{q}{r^4} = 1-\frac{r_0^4}{r^4}+\frac{r_0^3}{gr^4}\,. \tag{4.32}$$

It follows that there is no naked singularity if $gr_0 > 0$, even if q is negative. This is consistent with the fact that the total energy $E = \frac{1}{8}\pi^2 g^{-1} r_0^3(gr_0+4)$ is positive definite when we have $gr_0 > 0$. The thermodynamic quantities for these topological soliton solutions are given by

$$E = \frac{\pi^2 r_0^3(4+gr_0)}{8g}\,, \qquad J = \frac{\pi^2 r_0^4}{8g}\,, \qquad Q_1 = Q_2 = \frac{\pi^2 r_0^3}{4g}\,. \tag{4.33}$$

4.4 $E+3gJ+\sum Q_i=0$: Case B

This condition is satisfied by

$$e^{\delta_1+\delta_2}=1-\frac{2}{3ag}\,. \tag{4.34}$$

For simplicity and clarity, we first consider the case with two equal charges, namely $\delta_1=\delta_2$. Making the same changes of the variables and coordinates as in (4.20), we find that the solution becomes

$$\begin{aligned}
ds^2 &= H^{2/5}\Big(-\frac{V}{H^2B}r^2d\tilde t^2+B(\sigma+f\,d\tilde t)^2+\frac{dr^2}{V}+r^2d\Sigma_2^2\Big)\,,\\
A^1_{(1)} &= A^2_{(1)}=\frac{1}{r^4H}\frac{q+2g\alpha}{q-2g\alpha}(qd\tilde t+\alpha\sigma)\,,\qquad A_{(2)}=-\frac{\alpha}{r^4H}d\tilde t\wedge\sigma\,,\qquad A_{(3)}=-\frac{\alpha}{r^2}\sigma\wedge J\,,\\
X_1 &= X_2=H^{-1/5}\,,\qquad H=1+\frac{q}{r^4}\,,\\
V &= 1+g^2r^2H^2-\frac{2g\alpha(3q+2g\alpha)}{(q-2g\alpha)r^4}+\frac{8g\alpha^3}{(q-2g\alpha)^2r^6}\,,\\
B &= r^2-\frac{\alpha^2}{r^8H^2}\Big(1-\frac{8g\alpha}{(q-2g\alpha)^2}r^4\Big)\,,\qquad f=\frac{1}{B}\Big(gr^2+\frac{(q+2g\alpha)^2\alpha}{(q-2g\alpha)^2r^4H^2}\Big)\,.
\end{aligned} \tag{4.35}$$

From (4.9), the thermodynamic quantities are given by

$$\begin{aligned}
E &= \frac{\pi^2(4q^3-g\alpha q^2-4g^2\alpha^2q+4g^3\alpha^3)}{8(q-2g\alpha)^2}\,,\qquad J=-\frac{\pi^2\alpha(q^2+4g\alpha q-4g^2\alpha^2)}{8(q-2g\,\alpha)^2}\,,\\
Q_1 &= Q_2=-\frac{\pi^2(q+2g\alpha)(q-g\alpha)}{4(q-2g\alpha)}\,.
\end{aligned} \tag{4.36}$$

In the supersymmetric limit discussed in section 4.3, it was $K_+=\partial/\partial\tilde t$ that had the spinorial square root, corresponding to having an angular velocity $+g$ at infinity. By contrast, in the present case we find that the Killing vector with the spinorial square root is given by $K_-=\partial/\partial\tilde t-2g\partial/\partial\psi$, corresponding to having an angular velocity $-g$ at infinity. We then find

$$g(K_-,K_-)=H^{2/5}\Big(-\frac{V}{H^2B}r^2+B(f-2g)^2\Big)=-H^{-8/5}\Big(1-\frac{2g\alpha}{r^4}\Big)^2\,, \tag{4.37}$$

which is non-positive, consistent with the supersymmetry. Thus in general the horizon, where $V=0$, occurs when B is negative, implying the occurrence of naked CTC's.

However, in this case it is possible to arrange that the right-hand side of equation (4.37) is zero at $V=0$. (This is not possible for the previous case in (4.22).) Thus supersymmetric black holes without naked CTC's can arise in the present case. The requirement for such a solution can be easily obtained by requiring that $V(r_0)=0$ and $(1-2g\alpha/r_0^4)=0$ at some radius $r=r_0$. This implies that the parameters should be chosen so that

$$\alpha=\frac{r_0^4}{2g}\,,\qquad q^2=r_0^8+\frac{r_0^6}{g^2}\,. \tag{4.38}$$

With these choices, the metric function V is given by

$$V=\frac{g^2(r^2-r_0^2)^2}{r^2}\Big(1+\frac{r_0^8+q^2}{r_0^6r^2}+\frac{r_0^8+2qr_0^4+2q^2}{r_0^4r^4}\Big)\,, \tag{4.39}$$

which, remarkably, in fact has a *double* root at $r=r_0$. At $r=r_0$, we have that

$$B(r_0)=\tfrac{3}{4}r_0^2+\tfrac{1}{2}g^2(q+r_0^4)\,. \tag{4.40}$$

It follows from (4.38) that $q>r_0^4$, and so $B(r)$ is positive on the horizon. In fact it is straightforward to verify that $B(r)$ is positive definite at all radii from the horizon at $r=r_0$ to $r=\infty$. Thus the metric (4.35) with the parameters satisfying (4.38) describe a supersymmetric black hole that is regular everywhere on and outside the horizon, with the near-horizon geometry being the product of AdS$_2$ and a squashed S^5. The solution preserves $\frac{1}{8}$ of the supersymmetry. The double root of $V(r)$ at $r=r_0$ implies that the supersymmetric black hole has zero temperature. This seven-dimensional supersymmetric black hole is analogous to the five-dimensional one found in [23], which we discussed in section 3.4.

As in the five-dimensional case discussed in section 3.4, an alternative way to avoid CTC's is to consider the possibility that both V and B vanish at some radius $r=r_0$. This can be achieved by choosing the parameters so that

$$\begin{aligned} q &= \frac{g\alpha^3(1+4g^2r_0^2)+\alpha^2r_0^4(1-3g^2r_0^2)+g\alpha r_0^{10}}{r_0^2(3g^2\alpha^2-4g\alpha r_0^4+r_0^8}\,,\\ 0 &= g^2\alpha^4(1-2g^2r_0^2)^2+2g\alpha^3r_0^4(1+15g^2r_0^2+2g^4r_0^4)-\alpha^2r_0^8(-1+24g^2r_0^2+3g^4r_0^4)\\ &\quad +2g\alpha r_0^{14}(3-g^2r_0^2)+g^2r_0^{20}\,. \end{aligned} \tag{4.41}$$

Thus, for the cases where r_0 is the largest root for both V and B, the solution describes a smooth supersymmetric soliton.

The remaining solutions inevitably have naked CTC's. For those with a Killing horizon, *i.e.* $V=0$ at some $r=r_0>0$ outside the singularity at $r=0$, the $\tilde{t}$ coordinate must be appropriately periodically identified, in order to avoid a conical singularity at $r=r_0$. Having done so, the geodesics are then complete from $r=r_0$ to $r=\infty$. As usual, if V has a *double* root at $r=r_0$, then the $\tilde{t}$ coordinate does not require periodic identification. A double root is achieved by choosing the parameters to be given by

$$\alpha=-\frac{(16g^2r_0^2+9)(4g^2r_0^2+3)^2}{32g^5}\,,\qquad q=\frac{3(4g^2r_0^2)^2}{16g^4}\,. \tag{4.42}$$

The metric function V becomes

$$V=\frac{g^2(r^2-r_0^2)^2}{r^2}\Big(1+\frac{2g^2r_0^2}{g^2r^2}+\frac{72g^4r_0^4+88g^2r_0^2+27}{8g^2r^4}\Big) \tag{4.43}$$

It is straightforward to verify that at $r=r_0$, the function $B(r)$ is negative, implying naked CTC's. Thus the solution describes a supersymmetric BMPV type of repulson.

For non-equal charges, the solution becomes much complex, but the structure is very similar. The metric can be cast into the form

$$ds^2=(H_1H_2)^{\frac{1}{5}}\Big(-\frac{V}{H_1H_2B}r^2d\tilde{t}^2+B(\sigma+fd\tilde{t})^2+\frac{dr^2}{V}+r^2d\Sigma_2^2\Big)\,, \tag{4.44}$$

where $\tilde{t}=t/\Xi_-$ and $r=R/\sqrt{\Xi}$. The simplest way to determine if there exists a supersymmetric black hole is to examine the norm of the Killing vector $K_-=\partial/\partial_{\tilde{t}}-2g\,\partial/\partial_\psi$; it is given by

$$\begin{aligned} g(K_-,K_-) &= (H_1H_2)^{\frac{1}{5}}\Big(-\frac{Y}{f_1}+B^2(f-2g)^2\Big)\,,\\ &= -(H_1H_2)^{-4/5}\Big(1-\frac{162(e^{2\delta_1}-1)(e^{2\delta_2}-1)(e^{\delta_1+\delta_2}-1)^4}{e^{\delta_1+\delta_2}(3e^{\delta_1+\delta_2}-5)^2(3e^{\delta_1+\delta_2}-1)^3\,r^4}\Big)\,, \end{aligned} \tag{4.45}$$

which is non-positive. Thus naked CTC's can only be avoided in two circumstances. One is that the Killing horizon is coincident with the VLS, *i.e.* $V=0=B$ at some radius $r=r_0$. This leads to a regular supersymmetric soliton.

Alternatively, we can require that at $V=0$, the right-hand side of (4.45) also vanishes. This can be achieved if we take

$$m=\frac{128e^{\delta_1+\delta_2}(3e^{\delta_1+\delta_2}-1)^2}{729g^4(e^{2\delta_1}-1)(e^{2\delta_2}-1)(e^{\delta_1+\delta_2}+1)^2(e^{\delta_1+\delta_2}-1)^4}\,, \tag{4.46}$$

Then remarkably, the function V has a double root at $r=r_0$, given by

$$r_0^2=\frac{16}{3(e^{\delta_1+\delta_2}+1)(3e^{\delta_1+\delta_2}-5)g^2}\,. \tag{4.47}$$

With these choices of charge parameters, we find that the function V now becomes

$$V=\frac{g^2(r^2-r_0^2)^2}{r^2}\Big(1+\frac{9e^{2(\delta_1+\delta_2)}-6e^{\delta_1+\delta_2}+17}{3(e^{\delta_1+\delta_2}+1)(3e^{\delta_1+\delta_2}-5)g^2r^2}+\frac{h}{g^4r^4}\Big)\,, \tag{4.48}$$

where h is a constant, given by

$$\begin{aligned} h &= \Big[32(-2d_1^2-2d_2^2+9d_1d_2+9d_1^5d_2^5-3d_1^3d_2^3(d_1+d_2)^2+2d_1^2d_2^2(2d_1^2-3d_1d_2+2d_2^2)\\ &\quad -d_1d_2(3d_1^2-2d_1d_2+3d_2^2))\Big]/\Big[9d_1d_2(d_1^2-1)(d_2^2-1)(d_1d_2+1)(3d_1d_2-5)^2\Big]\,, \end{aligned} \tag{4.49}$$

with $d_1=e^{\delta_1}$ and $d_2=e^{\delta_2}$. The function $B(r)$ at r_0 is positive, and it is positive for all $r\ge r_0$, implying a supersymmetric black hole regular on and outside the horizon. This is the unequal-charge generalisation of the equal-charge regular black holes that we obtained above. These seven-dimensional supersymmetric black holes are analogous to the five-dimensional ones obtained in [24], which we discussed in section 3.4. Note that the right-hand side of (4.45) is negative definite if we turn off either of the charges, in which case there can be no supersymmetric black holes.

5 Rotating Black Holes and Supersymmetric Limits in Four-Dimensional Gauged Supergravity

5.1 Black-Hole Thermodynamics in Four Dimensions

Charged rotating black holes in four-dimensional Einstein-Maxwell theory with a cosmological constant were found in [15]. Recently, generalisations were obtained which can be viewed as charged rotating black holes in four-dimensional $\mathcal{N}=4$ gauged supergravity, with independent charges carried by the two gauge fields in the $U(1)\times U(1)$ abelian subgroup of the $SO(4)$ gauge group [18]. They can also, therefore, be viewed as solutions in $\mathcal{N}=8$ gauged supergravity, where the four charges associated with the $U(1)^4$ abelian subgroup of $SO(8)$ are set pairwise equal. The truncation of the $\mathcal{N}=4$ Lagrangian to the relevant sector for describing these solutions is given by

$$\begin{aligned}
\mathcal{L}_4 &= R*\mathbb{1}-\tfrac{1}{2}*d\varphi\wedge d\varphi-\tfrac{1}{2}e^{2\varphi}*d\chi\wedge d\chi-\tfrac{1}{2}e^{-\varphi}*F_{(2)2}\wedge F_{(2)2}-\tfrac{1}{2}\chi\,F_{(2)2}\wedge F_{(2)2}\\
&\quad -\frac{1}{2(1+\chi^2 e^{2\varphi})}\,(e^{\varphi}*F_{(2)}1\wedge F_{(2)1}-e^{2\varphi}\,\chi\,F_{(2)1}\wedge F_{(2)1})\\
&\quad -g^2\,(4+2\cosh\varphi+e^{\varphi}\,\chi^2)*\mathbb{1}\,.
\end{aligned}\tag{5.1}$$

The non-extremal rotating charged black hole solutions are given, in a frame that rotates at infinity, by [18]

$$\begin{aligned}
ds_4^2 &= -\frac{\Delta_r}{W}\,(dt-a\,\sin^2\theta\,\Xi^{-1}d\phi)^2+W\Big(\frac{dr^2}{\Delta_r}+\frac{d\theta^2}{\Delta_\theta}\Big)+\frac{\Delta_\theta\,\sin^2\theta}{W}[a dt-(r_1r_2+a^2)\Xi^{-1}d\phi]^2\,,\\
e^{\varphi_1} &= \frac{r_1^2+a^2\cos^2\theta}{W}=1+\frac{r_1\,(r_1-r_2)}{W}\,,\qquad \chi_1=\frac{a\,(r_2-r_1)\,\cos\theta}{r_1^2+a^2\,\cos^2\theta}\,,\\
A_{(1)1} &= \frac{2\sqrt{2}m\,s_1\,c_1\,r_2\,(dt-a\,\sin^2\theta\,\Xi^{-1}\,d\phi)}{W}\,,\\
A_{(1)2} &= \frac{2\sqrt{2}m\,s_2\,c_2\,r_1\,(dt-a\,\sin^2\theta\,\Xi^{-1}\,d\phi)}{W}\,,
\end{aligned}\tag{5.2}$$

where

$$\begin{aligned}
r_i &= r+2m\,s_i^2\,,\\
\Delta_r &\equiv \Delta+g^2\,r_1\,r_2\,(r_1\,r_2+a^2)=r^2+a^2-2m\,r+g^2\,r_1\,r_2\,(r_1\,r_2+a^2)\,,\\
\Delta_\theta &\equiv 1-g^2\,a^2\,\cos^2\theta\,,\qquad W=r_1\,r_2+a^2\cos^2\theta\,,\qquad \Xi=1-a^2\,g^2\,.
\end{aligned}\tag{5.3}$$

As usual, $s_i=\sinh\delta_i$ and $c_i=\cosh\delta_i$. Note that we are using the "undualised" form of the four-dimensional theory here, where, as discussed in [18], all the charges are electric. Note also that we have rescaled the azimuthal coordinate ϕ by a factor of Ξ^{-1} here, relative to the normalisation used in [18], so that ϕ in (5.2) has the canonical period 2π.

In order to give a more uniform treatment of these four-dimensional solutions that harmonises with our discussion in five and seven dimensions, we shall adopt a maximal supergravity notation at this point, and view the solution (5.2) as a 4-charge solution with pairwise equal charges. This will avoid the necessity for $\sqrt{2}$ factors associated with the charges. Thus we shall have charges $Q_1 = Q_2$ characterised by the parameter δ_1, and charges $Q_3 = Q_4$ characterised by δ_2. This change of viewpoint will be understood in all our subsequent formulae for charges and electrostatic potentials.

The coordinates in (5.2) are rotating at infinity. A non-rotating coordinate system is achieved by defining a new azimuthal angle $\phi' = \phi + a\, g^2\, t$. The time coordinate t has the canonical normalisation. It is helpful to recast the metric (5.2) in the form

$$ds_4^2 = -\frac{\Delta_r\,\Delta_\theta}{B\Xi^2}\,dt^2 + B\,\sin^2\theta\,(d\phi + f\,dt)^2 + W\Big(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\Big)\,, \tag{5.4}$$

The entropy, temperature, angular velocity and electrostatic potentials on the horizon are given by

$$\begin{aligned} S &= \frac{\pi(r_1 r_2 + a^2)}{\Xi}\,, \qquad T = \frac{\Delta_r'}{4\pi(r_1 r_2 + a^2)}\,, \qquad \Omega = \frac{a(1 + g^2\, r_1 r_2)}{r_1 r_2 + a^2}\,,\\ \Phi_1 &= \Phi_2 = \frac{2m s_1 c_1\, r_2}{r_1 r_2 + a^2}\,,\\ \Phi_3 &= \Phi_4 = \frac{2m s_2 c_2\, r_1}{r_1 r_2 + a^2}\,, \end{aligned} \tag{5.5}$$

where all quantities are evaluated on the outer horizon at $r = r_+$, the largest root of Δ_r. Calculating the angular momentum, and the charges, as surface integrals at infinity, we can then integrate the first law

$$dE = T\,dS + \Omega\,dJ + \sum_i \Phi_i\,dQ_i\,, \tag{5.6}$$

to obtain the energy. Our results are

$$\begin{aligned} E &= \frac{m}{\Xi^2}\,(1 + s_1^2 + s_2^2) = \frac{m}{2\Xi^2}\,(\cosh 2\delta_1 + \cosh 2\delta_2)\,,\\ J &= \frac{m\,a}{\Xi^2}\,(1 + s_1^2 + s_2^2) = \frac{ma}{2\Xi^2}\,(\cosh 2\delta_1 + \cosh 2\delta_2)\,,\\ Q_1 = Q_2 &= \frac{m\,s_1\,c_1}{2\Xi} = \frac{m}{4\Xi}\,\sinh 2\delta_1\,,\\ Q_3 = Q_4 &= \frac{m\,s_2\,c_2}{2\Xi} = \frac{m}{4\Xi}\,\sinh 2\delta_2\,. \end{aligned} \tag{5.7}$$

5.2 Supersymmetric Limits and the Supersymmetric Bound

The algebra of the supercharges $\mathcal{Q}$ in four-dimensional AdS supergravity is given by

$$M \equiv \{\mathcal{Q}, \overline{\mathcal{Q}}\} = \tfrac{1}{2} J_{AB}\,\gamma^{AB} + Z\,, \tag{5.8}$$

where J_{04} is the energy, and J_{ij} for $1 \le i \le 3$ correspond to angular momenta. Taking $J_{12} = J$ non-zero, we shall have

$$J_{04} = g\, E\,, \qquad J_{12} = g^2\, J\,, \qquad Z = g\, \sum_i Q_i\,, \tag{5.9}$$

after using the adjoint action of the AdS and R-charge symmetries to choose a convenient frame for Z^{IJ}. The four eigenvalues of the Bogomolny matrix $g^{-1}\, M_\alpha{}^\beta$ are given by

$$\lambda = E \pm g\, J \pm \sum_i Q_i\,. \tag{5.10}$$

The four eigenvalues are equivalent, modulo sign reversals of the angular momentum and the charges, so unlike the $D = 5$ and $D = 7$ cases discussed previously, there is only one inequivalent case of interest to consider here.

The general analysis of the convex cone implied by the positivity of the Bogomolny matrix eigenvalues described in section 3.2 applies to this $D = 4$ case as well. There is just one angular momentum and so the cone lies in $\mathbb{R}^3$ and is bounded by two planes.

Substituting our results (5.7) for the energy, angular momentum and charges of the rotating black holes (5.2) into (5.10), we find that a zero eigenvalue is achieved if

$$e^{2\delta_1+2\delta_2} = 1 + \frac{1}{2ag}\,. \tag{5.11}$$

The solution preserves $\frac{1}{4}$ of the supersymmetry.

To discuss the global structure of the solution, we can examine the metric function Δ_r, which after imposing the supersymmetry condition, can be expressed as a sum of two squares:

$$\begin{aligned} \Delta_r &= g^2\Big(r^2 + m(\sinh^2\delta_1 + \sinh^2\delta_2)\, r + 4m^2 \sinh^2\delta_1\, \sinh^2\delta_2 + g^{-2}(\coth(\delta_1+\delta_2) - 1)\Big)^2 \\ &\quad + \tanh^2(\delta_1+\delta_2)\Big(r - \frac{2m \sinh\delta_1\, \sinh\delta_2}{\cosh(\delta_1+\delta_2)}\Big)^2\,. \end{aligned} \tag{5.12}$$

Thus in general the function Δ_r has no root, and hence the solution has a naked singularity. The only possible root is given by

$$r_+ = \frac{2m \sinh\delta_1\, \sinh\delta_2}{\cosh(\delta_1+\delta_2)}\,, \tag{5.13}$$

which is achieved by taking

$$m\, g = \frac{\cosh(\delta_1+\delta_2)}{e^{\frac{1}{2}(\delta_2+\delta_2)} \sinh^2(\delta_1+\delta_2) \sinh(2\delta_1) \sinh(2\delta_2)}\,. \tag{5.14}$$

The function Δ_r then has a double root r_+, and hence the solution describe a supersymmetric black hole that is regular on and outside the horizon, with zero temperature. In the case where the charge parameters are set equal, this solution reduces to the regular supersymmetric AdS black hole found in [20].

6 Gödel Black Holes

In this section we shall apply some of the ideas of this paper to a closely related set of solutions of ungauged supergravity that are of some current interest, namely Gödel black holes [43, 44, 45, 46, 47]. Although the solutions themselves are not new and have received some considerable discussion, the particular point we are making appears to have been missed in the literature.

The Gödel black hole solutions of the ungauged 5-dimensional supergravity theory are given by

$$\begin{aligned} ds^2 &= -\frac{\Delta}{r^4\,\beta}\,dt^2+\tfrac14\beta r^2\Big(\sigma_3-\frac{4(am+br^4)}{r^4\beta}\,dt\Big)^2+\frac{r^4dr^2}{\Delta}+\tfrac14 r^2(\sigma_1^2+\sigma_2^2)\,,\\ A &= \frac{\sqrt3}{2}\,br^2\sigma_3\,, \end{aligned} \tag{6.1}$$

with

$$\Delta = r^4-(2m-16b^2m^2-8mab)r^2+2ma^2\,,\qquad \beta = 1-8mb^2+\frac{2ma^2}{r^4}-4b^2r^2\,. \tag{6.2}$$

If m is set to zero, we obtain the pure Gödel background, whose spatial sections are flat. We shall discuss this metric in more detailo in section 6.1.

Expressions for the the energy E , angular momentum J, and charge Q have been obtained in [48]:

$$E = \frac{3\pi}{4}m-8\pi b^2m^2-\pi abm\,, \tag{6.3}$$

$$J = \tfrac12\pi ma-\pi ma^2b-4\pi b^2m^2a \tag{6.4}$$

$$Q = 2\sqrt3\,\pi mab\,. \tag{6.5}$$

The solution is singular at $r=0$; both the curvature and the field strength diverge there. The latter can be seen from

$$F^2 = \frac{48b^2(r^2-m)}{r^2}\,. \tag{6.6}$$

Note that the gauge field is magnetic at large distances, but becomes electric for $r^2<m$.

Since $g_{tt} = 2m/r^2-1$, there is an ergo-region when $r^2<2m$. Since

$$g_{\psi\psi} = \frac{r^2}{4}\beta(r)\,, \tag{6.7}$$

there is a VLS, situated at $r=r_L$, where

$$\beta(r_L) = 0\,. \tag{6.8}$$

If $a=0$, the VLS is located at $r^2=r_L^2=(1-8mb^2)/(4b^2)$. There are no CTC's within the domain $r<r_L$, and so the time machine occupies the region $r>r_L$. The Killing horizon is situated at $r^2=r_H^2=2m(1-8mb^2)$, which is always inside the ergo-region. One has

$$r_L^2-r_H^2=\frac{(1-8mb^2)^2}{4b^2}\,. \tag{6.9}$$

By contrast, if $a\neq 0$ the situation is more complicated. The horizon is now situated at

$$r_H^2=m-4mab-8b^2m^2+m\sqrt{\alpha}\,, \tag{6.10}$$

with

$$\alpha=(1-8b^2m)(1-8b^2m-8ab-\frac{2a^2}{m})\,. \tag{6.11}$$

One has entropy, angular velocity and surface gravity given by

$$\begin{aligned} S &= \tfrac{1}{2}\pi^2 r_H^3\sqrt{\beta(r_H)}\,,\\ \Omega &= \frac{4br_H^2+4ma}{r_H^2\,\beta(r_H)}\,,\\ \kappa &= \frac{2m\sqrt{\alpha}}{r_H^3\sqrt{\beta(r_H)}}\,. \end{aligned} \tag{6.12}$$

It was verified in [48] that the first law of thermodynamics holds for these expressions. Thus the usual thermodynamic interpretation holds, as long as the black hole lies in inside the region where there are no CTC's.

One may instead consider the case where the horizon moves into the time machine, *i.e.* $\beta(r_H)<0$ and thus $r_H>r_L$. In this case, the horizon is really to be thought of as a pseudo-horizon that closes off the spacetime. Thus the radial coordinate cannot exceed r_H, and at $r=0$ there is a naked singularity. The surface gravity is now purely imaginary, and so the (real) time t must be identified with the real period

$$\Delta t=\frac{2\pi r_H^3\sqrt{|\beta(r_H)|}}{2m\sqrt{\alpha}}\,. \tag{6.13}$$

A point apparently missed in the literature on Gödel black holes is that the gauge field given in (6.1) becomes singular at $r=r_H$, which may be seen from the fact that

$$A_\mu\,A_\nu\,g^{\mu\nu}=\frac{3b^2r^4(r^2-2m)}{\Delta}\,. \tag{6.14}$$

If $r_H<r_L$ we may pass to a new gauge potential

$$A'=A-\frac{2\sqrt{3}b(am+br_H^4)}{r_H^2\,\beta(r_H)}\,dt\,, \tag{6.15}$$

for which $|A'|^2 = P(r)/\Delta(r)$, where $P(r)$ is a poynomial of degree three in r^2 which vanishes at $r = r_H$. If $r_H < r_L$ and if r_H is the largest positive root of Δ, then the gauge potential A' will be regular everywhere outside and on the horizon. If $r_H > r_L$, and the next-smallest root r_- of Δ is smaller than r_L, the gauge potential is bounded for $r_- < r \le r_H$. However, because we now have a pseudo-horizon, for which the coordinate t must be identified with period given by (6.13), we obtain a Josephson quantisation condition of the form

$$\frac{\sqrt{3}\, eb(am+br_H^4)r_H}{m\sqrt{\alpha\,\beta(r_H)}}\,, \tag{6.16}$$

if there are fields of charge e present.

In ungauged supergravity, the Dirac or Josephson quantisation conditions need not hold, since the fields in the supergravity multiplet are uncharged. There does also exist a five-dimensional Gödel-AdS type solution in gauged supergravity, which was found in [49] and is given by

$$\begin{aligned} ds^2 &= -(dt+\omega)^2 + \frac{dr^2}{1+g^2r^2} + \frac{r^2}{4}\Big(\sigma_1^2+\sigma_2^2+(1+g^2r^2)\sigma_3^2\Big)\,,\\ \omega &= \frac{r^2}{2}(g\sigma_3 - \frac{h}{1+g^2r^2}\sigma_1)\,,\\ A &= \frac{\sqrt{3}hr^2}{2(1+g^2r^2)}\sigma_1\,. \end{aligned} \tag{6.17}$$

The field strength has norm given by

$$F_{\mu\nu}\,F^{\mu\nu} = \frac{48h^2}{(1+g^2r^2)^3}\,. \tag{6.18}$$

If $h = 0$, we get the manifestly $U(2)\times\mathbb{R}$ invariant AdS$_5$ metric with a Bergmann base. If $h \ne 0$ the manifest $U(2)$ is broken to $SU(2)$. If instead g is set to zero, we obtain the pure Gödel background, which will be discussed in section 6.1.

The metric induced on the $SU(2)$ orbits, which are squashed at infinity, is

$$\frac{r^2}{4}\Big(\sigma_2^2 + (\sigma_3 - \frac{ghr^2}{1+g^2r^2}\,\sigma_1)^2 + \frac{1+(g^2-h^2)r^2}{1+g^2r^2}\,\sigma_1^2\Big)\,. \tag{6.19}$$

Thus the orbits of $SU(2)$ remain spacelike at large r, or become timelike, depending on whether

$$g^2 > h^2\,, \qquad \text{or} \qquad h^2 > g^2 \tag{6.20}$$

respectively.

In both cases the coordinate t is globally defined, and there is no need to make an identification. In fact

$$g^{tt} = g^{\mu\nu}\partial_\mu t\,\partial_\nu t = -\frac{[1+(g^2-h^2)r^2]}{(1+g^2r^2)^2}\,. \tag{6.21}$$

In the first case, $\partial_\mu t$ is always timelike, and therefore the coordinate t is a global time function, *i.e.* it increases along every future-directed timelike curve, and there are no CTC's. In the latter case, $\partial_\mu t$ ceases to be timelike outside a VLS, which is located at

$$r^2 = \frac{1}{h^2 - g^2}\,. \tag{6.22}$$

Thus in this case, while being a globally-defined coordinate, t is not a global time function. The norm of the potential A given in (6.17) is

$$A_\mu\, A_\nu\, g^{\mu\nu} = \frac{3h^2 r^2}{(1+g^2r^2)^2}\,, \tag{6.23}$$

which is regular everywhere. We conclude that since A is globally defined, there is no need to impose any quantisation condition. One might wonder whether, if one chose to identify t periodically, a quantisation condition would result. However, because the potential A is globally well-defined even if t is periodically identified, no quantisation condition would arise in this case either. Curiously, there exist two globally well-defined gauges in which the new gauge potentials A' fall off faster at infinity than does A, which falls of like $1/r^2$. Namely, if we define

$$A' = A - \frac{\sqrt{3}\,h}{h \pm g}\, dt\,, \tag{6.24}$$

then we find

$$A'_\mu\, A'_\nu\, g^{\mu\nu} = -\frac{3h^2}{(h\pm g)^2\,(1+g^2r^2)^2}\,. \tag{6.25}$$

Since both of these gauges are globally well-defined, no quantisation condition would arise, even if we insisted upon the faster fall-off that they exhibit.

This analysis of transition functions is supported by the observation that the topology of the solution (6.17) is trivial; it is the product, topologically, of the time $\mathbb{R}$ and the Bergmann manifold, which itself is topologically $\mathbb{R}^4$.

6.1 Heisenberg Quantization Conditions

While on the subject of quantisation conditions, it may be of interest to reconsider the pure Gödel solution. This is the ground state with respect to which the energy, angular momentum and charge of the Gödel black holes are measured. Since one must pass through the VLS in order to travel backwards in time, one might consider compactifying the spatial sections so that a unit cell lies inside the VLS, in order to prevent time travel. It turns out that one may indeed compactify the spatial sections, but only at the expense of passing to a periodic time coordinate.

To see this, note that the pure Gödel ground state metric may be cast in the form

$$ds^2 = -[dt+2b(xdy-ydx+zdw-wdz)]^2+dx^2+dy^2+dz^2+dw^2\,. \tag{6.26}$$

The metric is homogeneous, since it admits the five Killing fields $R_t = \partial_t$ and

$$R_x = \partial_x - 2b\,y\partial_t \qquad R_y = \partial_y + 2b\,x\,\partial_t \qquad R_z = \partial_z - 2b\,w\partial_t \qquad R_w = \partial_w + 2b\,z\partial_t \tag{6.27}$$

One checks that time translation is central, and moreover, the only non-vanishing brackets are

$$[R_x, R_y] = 4bR_t \qquad [R_z, R_w] = 4bR_t\,. \tag{6.28}$$

This gives a Heisenberg type algebra $\mathfrak{g}$. In fact one may regard the metric as a left-invariant metric on the Heisenberg type group G.

The one forms $dt+2b(xy-ydx+zdw-wdz)$ and dx, dy, dz, dw are left-invariant. The vector fields R_t, R_x, R_y, R_z, R_w generate left translations, and are right-invariant.

The gauge field supporting the metric,

$$A = \frac{\sqrt{3}}{2}(dt+2b(xdy-ydx+zdw-wdz)) - \frac{\sqrt{3}}{2}dt\,, \tag{6.29}$$

is invariant only up to a time-dependent gauge transformation because

$$\pounds_{R_x} dt = -dy\,, \qquad \pounds_{R_y} dt = dx\,. \tag{6.30}$$

The gauge potential (6.29) satisfies

$$A_\mu\, A_\nu\, g^{\mu\nu} = 3b^2\,(x^2+y^2+z^2+w^2)\,. \tag{6.31}$$

We note that the norm of the field strength is given by

$$F_{\mu\nu}\, F^{\mu\nu} = 48b^2\,. \tag{6.32}$$

One could pass to a new gauge, in which the transformed gauge potential $\tilde{A}$ is invariant, by making the gauge transformation

$$A \to \tilde{A} = A + d(\frac{\sqrt{3}}{2}t)\,. \tag{6.33}$$

The new gauge field $\tilde{A}$ has constant magnitude

$$\tilde{A}_\mu\, \tilde{A}_\nu\, g^{\mu\nu} = -\frac{3}{4}\,. \tag{6.34}$$

Acting on a field Ψ of charge e, the necessary gauge transformation is

$$\Psi \to e^{(ie\frac{\sqrt{3}}{2})t}\Psi\,. \tag{6.35}$$

In order to discuss the identifications it is helpful to introduce some convenient notation. If V is a vector field, the operator $e^{\lambda V}$ acting on functions gives

$$e^{\lambda V} f(x^\mu) = f(\tilde{x}^\mu)\,, \tag{6.36}$$

where $\tilde{x}$ is the point obtained by moving a parameter distance λ along the integral curves of V. In other words $\tilde{x}^\mu = x^\mu(\lambda)$, where

$$\frac{dx^\mu}{d\lambda} = V^\mu(x)\,, \qquad x^\mu(0) = x^\mu\,. \tag{6.37}$$

We could write

$$\tilde{x}^\mu = e^{\lambda V} x^\mu\,. \tag{6.38}$$

Thus, for example,

$$e^{\lambda R_x} f(t,x,y,z,w) \;=\; f(t - 2by\lambda, x + \lambda, y, z, w)\,, \tag{6.39}$$

$$e^{\lambda R_y} f(t,x,y,z,w) \;=\; f(t + 2bx\lambda, x, y + \lambda, z, w)\,. \tag{6.40}$$

If ϕ_λ is the one-parameter group of diffeomorphisms associated to the vector field V, the usual definition of pull-back becomes in this notation

$$\phi^\star_\lambda f(x) = f(\phi^{-1}_\lambda(x)) = e^{-\lambda V} f(x)\,. \tag{6.41}$$

Now consider attempting to identify the x coordinate, with period d_1 say. As it stands, this is not a symmetry, and therefore one must shift in time as well, *i.e.* we demand that

$$(t,x,y,z,w) \equiv (t - 2byd_1, x + d_1, y, z, w)\,. \tag{6.42}$$

If we instead identify the y coordinate, with period d_2, we must demand that

$$(t,x,y,z,w) \equiv (t + 2bxd_2, x, y + d_2, z, w)\,. \tag{6.43}$$

However, these two identifications do not commute. In fact, one easily checks from the Lie algebra that

$$e^{d_1 R_x} e^{d_2 R_y} e^{-d_1 R_x} e^{-d_2 R_y} = e^{4 d_1 d_2 b R_t}\,. \tag{6.44}$$

One must therefore identify the time coordinate as well, *i.e.* demand that

$$(t,x,y,z,w) \equiv (t + 4bd_1 d_2, x, y, z, w)\,. \tag{6.45}$$

Similar considerations apply if one wishes to identify z and w, with periods d_3 and d_4 say. This also entails an identification of t, with period $4bd_3d_4$. Consistency then requires that

$$d_1 d_2 = d_3 d_4\,. \tag{6.46}$$

7 Conclusions

In this paper, we have studied the thermodynamics of the recently-discovered non-extremal charged rotating black holes of gauged supergravities in five, seven and four dimensions, obtaining energies, angular momenta and charges that are consistent with the first law of thermodynamics. We studied their supersymmetric limits, by using these expressions together with a Bogomolny analysis of the AdS superalgebras. We gave a general discussion of the global structure of such solutions, and applied it in the various cases. We obtained new regular supersymmetric black holes in seven and four dimensions, as well as reproducing known examples in five and four dimensions. We also obtained new supersymmetric non-singular topological solitons in five and seven dimensions. The rest of the supersymmetric solutions either have naked singularities or naked time machines. The latter can be rendered non-singular if the asymptotic time is periodic. This leads to a new type of quantum consistency condition, which we call a *Josephson quantisation condition*. Finally, we discussed some aspects of rotating black holes in Gödel universe backgrounds.

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A The Spin^c Structure of the Taub-BOLT Manifold

In this appendix, we give a brief discussion of the spin^c structure of the $k = 1$ $\mathbb{R}^2$ bundle over S^2, which is the manifold of the Taub-BOLT instanton found in [36]. This serves as a useful illustrative example that exhibits some of the same essential features that arise in the topological soliton solutions of section 3.3.1 in the case that the integer k is odd.

The Ricci-flat Taub-BOLT instanton metric is given by [36]

$$ds^2 = \frac{(r^2-\ell^2)\,dr^2}{(r-2\ell)(r-\frac{1}{2}\ell)} + \frac{4\ell^2\,(r-2\ell)(r-\frac{1}{2}\ell)}{r^2-\ell^2}\,\sigma_3^2 + (r^2-\ell^2)\,(\sigma_1^2+\sigma_2^2)\,, \tag{A.1}$$

where $r \ge 2\ell$, and the Euler angles in the $SU(2)$ left-invariant 1-forms have the standard periods, $0 \le \phi < 2\pi$, $0 \le \psi < 4\pi$, and $0 \le \theta \le \pi$. There are two L^2 harmonic forms, which are self-dual and anti-self-dual respectively, and given locally by $F_i = dA_i$, where the potentials are

$$A_1 = \Big(\frac{r+\ell}{r-\ell}\Big)\,\sigma_3\,, \qquad A_2 = \Big(\frac{r-\ell}{r+\ell}\Big)\,\sigma_3\,. \tag{A.2}$$

In fact the combination

$$\tfrac{1}{8}(9A_2 - A_1) = \frac{(r-2\ell)(r-\frac{1}{2}\ell)}{r^2-\ell^2}\,\sigma_3 \tag{A.3}$$

is globally defined, and the associated field strength is exact. The combination

$$A = -\tfrac{3}{8}P\,(A_1 - A_2) = -\frac{3P\,\ell\,r}{2(r^2-\ell^2)}\,\sigma_3\,, \tag{A.4}$$

which falls off at large r and is singular on the Bolt at $r = 2\ell$, defines a regular 2-form $F = dA$ whose integrals over the S^2 bolt and the $\mathbb{R}^2$ bundle parameterised by (r,ψ) are given by

$$\frac{1}{4\pi}\int_{S^2} F = P\,, \qquad \frac{1}{4\pi}\int_{\mathbb{R}^2} F = P\,. \tag{A.5}$$

If there are fermions with charge e, the usual Dirac quantisation conditions would imply that $2eP$ should be an integer. However, since the Taub-BOLT manifold does not admit a spin structure, consistency of the fermion wave functions requires that instead, as discussed in [38], we impose the quantisation condition

$$2eP = q + \tfrac{1}{2}\,, \tag{A.6}$$

where q is an integer.

A consistency check, analogous to the one described in [38] for $\mathbb{CP}^2$, can be performed by calculating the Atiyah-Singer index for the Dirac operator for such charged spinors in the Taub-BOLT manifold. Thus, the difference between the numbers of right-handed and left-handed L^2-normalisable zero modes of the charged Dirac operator is given by

$$n_+ - n_- = -\frac{1}{384\pi^2}\int R_{\mu\nu\rho\sigma}\,{}^*R^{\mu\nu\rho\sigma}\,\sqrt{g}d^4x + \frac{e^2}{16\pi^2}\int F_{\mu\nu}\,{}^*F^{\mu\nu}\,\sqrt{g}d^4x - \tfrac{1}{2}\eta(0) \tag{A.7}$$

where $-\frac{1}{2}\eta(0) = -1/12$ is the Atiyah-Patodi-Singer term calculated from the eigenvalues of the Dirac operator on the boundary. From (A.1) and the field $F = dA$ coming from (A.4), we find that the Dirac index is given by

$$n_+ - n_- = 2e^2P^2 - \tfrac{1}{8}\,. \tag{A.8}$$

Thus we see that with the quantisation condition (A.6) appropriate to this case where there is no ordinary spin structure, the index is

$$n_+ - n_- = \tfrac{1}{2} q(q+1)\,, \tag{A.9}$$

which is indeed always an integer.

One can also check the Hirzebruch index of the operator $d+\delta$, which gives the difference between the numbers of self-dual and anti-self-dual harmonic 2-forms. For the self-dual Taub-NUT instanton, it is known that this index is -1, coming from a $-2/3$ contribution from the bulk integral

$$\frac{1}{48\pi^2} \int R_{\mu\nu\rho\sigma}\, {}^*R^{\mu\nu\rho\sigma}\, \sqrt{g} d^4x\,, \tag{A.10}$$

and a $-1/3$ from the boundary. The same boundary term arises for Taub-BOLT, and thus evaluating the contribution (A.10) for (A.1) we obtain the Hirzebruch signature $1/3-1/3 = 0$ for the Taub-BOLT metric. This is consistent with the existence of the one self-dual and one anti-self-dual L^2 harmonic forms that we found above.

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